A Study on Normal Gamma Seminear Rings

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Abstract:-

The concept gamma near rings was introduced by Bhavanarisathayanarayana and studied by several authors like Young ukcho, R. Balakrishnan. It was later extened to seminear ring and gamma seminear ring by R. Perumal. et. al. In this paper we derived some results on left (right) normal gamma seminear ring by using the concepts on regularity and idempotent elements. In a regular normal gamma seminear ring, it is proved that both aγb and bγa are idempotents. A left normal gamma seminear ring is regular if and only if aΓR = eΓR and it is right normal whenever e is idempotent. The definition of left (right) – r – normal gamma seminear ring is given in this paper along with the property that every left – r – normal is left normal and homomophic image is preserving the left normal property if r = 2. Every left ideal of left (right) normal gamma seminear ring is idempotent also on a particular case when r = 2.

Keywords:-

Gamma seminear rings, left and right normal, idempotent, regular, left – r – normal, right – r –normal.

1. Introduction:-

The concept of Seminear Rings was introduced by Willy G. V. Hoorn and B. V. Rootselaar in 1967 [19]. Seminear Rings are a common generalization of near rings and semi rings. However in [19] only a very special type of Seminear Rings was considered and question arose whether it is possible to develop a more general theory of Seminear Rings.

The concept of Γ – near ring was introduced by Sathyanarayana [3]. Also M. K. Rao [9] studied Γ – Semi ring and then N. K. Saha at el [10] defined the generalization of Γ Seminear Rings and studied its properties.

The purpose of this paper is to establish the concept of normal gamma seminear rings and obtain some of their properties. As a results seminear rings came into being as a common generalization of near rings and semi rings.

In section 2, we give preliminaries of gamma seminear rings which are used in the subsequent sections. In this section 3, we discuss the properties of right (left) normal, idempotent, left – 2- normal and in this full paper Normal Gamma Seminear Rings satisfying some properties.

Notation:-

We furnish below the notations that we make use of throughout this paper.

(1) E = $\{e \in R / e\gamma e = e\}$ – set of all idempotent of R.

(2) L = $\{x \in R / \{x\}^k = 0$ for some positive integer k$\}$ – set of all nilpotent element of R.

(3) C(R) = $\{r \in R / rx = xr$, for all $x \in R\}$.

2. Preliminaries:-

In this section we list some basic definitions and results from the theory of ΓSeminear rings that are used in the development of the paper.

2.1 Definition:-[1]

A non empty set N together with two binary operations '+' and '.' Satisfying the following conditions, is said to be *seminear rings*.

(a) (N, +) is a semigroup,
(b) (N, .) is a semigroup,
(c) $(x + y).z = x.z+x.y$ for all $x, y, z \in$ N.

Precisely speaking 'Seminear ring' is a right seminear ring' here since every seminear ring satisfy one distributive law (left / right distributive law.

Every near rings is a seminear ring but every seminear ring need not be a near ring. For this we consider the following example.

2.2 Example:-[1]

Let N = $\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} / a, b$ be non negative integers$\}$,

{N, +, .} is seminear ring under the matrix addition and matrix multiplication.

Here N is a seminear ring which is not a near ring since (N, +) is a semigroup but not a group since additive inverse does not exist for all members of N.

2.3 Definition:- [1]

Ler M be an additive semigroup and Γ a non-empty set.

Then M is called a *right Γ-seminear ring*, if there exists a mapping M×Γ×M → M satisfying the following conditions

a) $(a+b)\gamma c = a\gamma c + b\gamma c$,
b) $(a\gamma b)\beta c = a\gamma(b\beta c)$, for all $a,b,c \in M$ and $\gamma, \beta \in \Gamma$.

Precisely speaking 'Γ – seminear ring' to mean right Γ – seminear ring.

Every Γ– near rings is a Γ – seminear rings but every Γ – seminear ring need not be a Γ- near ring. For this we consider the following Example.

2.4 Example:-[1]

Let M = $\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} / a, b$ be non negative integers$\} = \Gamma$,

Then (M, +, Γ) is Γ – seminear ring under the matrix addition and matrix multiplication.

Here M is a seminear ring which is not a near ring since (M, +) is a semigroup but not a group since additive inverse does not exist for all members of M.

Define M $\times \Gamma \times M \to M$ (denoted by $(a, \alpha, b) \to a\alpha b$) where $a\alpha b$ is matrix multiplication of a, α, b

Then M is a Γ seminear ring but not Γnear ring. Since (M, +) is a semigroup which is not a group.

2.5 Definition:-[1]

Let M be a Γ seminear ring. A non empty subset M'of M is a *sub Γ – seminear ring* of M if M' is also a Γ- seminnear ring with the same operations of M.

2.6 Definition:-[4]

(R, +. Γ) is a right *Γ – seminear field* if,

(1) (R, +) is a semigroup.

(2) (R^*, Γ) is a group (R^* is R without addition zero, if it has one).

(3) $(a + b)\gamma c = a\gamma c + b\gamma c$, for all $a, b, c \in$ R, $\gamma \in \Gamma$.

2.7 Definition:-[4]

A *Γ – Seminear rings homomorphism* between two right Γ Seminear ring R and R' is map f : R -> R' satisfying

(1) $f(a + b) = f(a) + f(b)$,

(2) $f(a\gamma b) = f(a)\gamma f(b)$, for all $a, b \in$ R, $\gamma \in \Gamma$.

2.8 Definition:-[1]

A non-empty subset I of a Γ Seminear ring M is called *left (right) ideal*. If

a) For all $x, y \in I, x + y \in I$ and
b) For all $x \in I$, $a \in$ M and $\gamma \in \Gamma$, $x\gamma a$ $(a\gamma x) \in I$

I is said to be an ideal of M it is both a left and a right ideal.

2.9 Definition:-[5]

A Γ Seminear ring M will be called *regular*, if for every $a \in M$ there exists $b \in M$ such that $a = a\gamma_1 b\gamma_2 a$, for all $\gamma_1, \gamma_2 \in \Gamma$.

2.10 Definition:-[4]

A Γ Seminear ring M will be called *Boolean* if $x\gamma x = x$ for all x $\in M$ $\gamma \in \Gamma$.

.2.11 Definition:-[5]

A Γ Seminear ring M is called a *left (right) normal* Γ Seminear ring, if $a \in M\Gamma a$ $(a \in a\Gamma M)$ for each $a \in M$. The Γ Seminear ring M is normal if it is both left normal and right normal.

3. Normal Gamma SeminearRings:-

In this section, we dicuss the concept of left and right normal Γ Seminear rings.

3.1 Theorem:-

Homomorphic images of a left normal (right normal) Γ Seminear ring normal is also left (right).

Proof:-

Let M be a left normal Γ Seminear ring. Let f : M → M' be a Γ seminear ring epimorphism. As M is a left normal, we have $a \in M\Gamma a$. Now $a' = f(a) \in f(za)$ (for some $z \in M$) $= f(z)f(a) = z'a' \in M'a'$. The desired result now follows. The proof is similar for right normal Γ Seminear ring.

3.2 Proposition:-

Let M be a regular Γ Seminear ring. Then for every $a \in M$.

a) $a\gamma b$ and $b\gamma a$ are idempotent.
b) $Ma = Me$ and $aM = e'M$ for some idempotent e and e'

Proof:-

a) Since M is regular, for every $a \in M$, there exists $b \in M$ such that $a = a\gamma_1 b\gamma_2 a$ for all $\gamma_1, \gamma_2 \in \Gamma$. Now $(a\gamma b)^2 = (a\gamma_1 b)\gamma_2(a\gamma_3 b) = (a\gamma_1 b\gamma_2 a)\gamma_3 b = a\gamma_3 b$, for all $\gamma_1,\gamma_2,\gamma_3 \in \Gamma$. $(b\gamma a)^2 = (b\gamma_1 a)\gamma_2(b\gamma_3 a) = b\gamma_1(ba\gamma_2 b\gamma_3 a) = b\gamma_1 a$, for all $\gamma_1,\gamma_2,\gamma_3 \in \Gamma$. Hence Proved $a\gamma b$ and $b\gamma a$ are idempotent.

b) We observe that $M\Gamma a = M\Gamma a\gamma_1 b\gamma_2 a \subseteq M\Gamma b\gamma_2 a \Rightarrow M\Gamma a \subseteq M\Gamma b\gamma_2 a$. Since $M\Gamma b\gamma_2 a \subseteq M\Gamma a$. We get $M\Gamma a = M\Gamma b\gamma_2 a = M\Gamma e$, where $e = b\gamma_2 a$ is an idempotent.

Similarly, $a\Gamma M = a\gamma_1 b\gamma_2 a\Gamma M \subseteq a\gamma_1 b\Gamma M \Rightarrow a\Gamma M = a\gamma_1 b\Gamma M = e'\Gamma M$, Where $e' = a\gamma_1 b$ is an idempotent. Hence Proved $Ma = Me$ and $aM = e'M$

3.3 Theorem:-

Let M be a Γ Seminear ring. Then M is regular if and only if M has the condition for all a ∈M there exists $e\gamma e = e \in M$ such that MΓa = MΓe and M is left normal.

Proof:-

Suppose that M is regular. Then for any $a \in M$ there exists $b \in M$ such that $a = a\gamma_1 b\gamma_2 a$. Since from proposition 3.2, that $R\Gamma a = R\Gamma e$. ObviouslyM is left normal.

Conversely, assume that M has the give condition, for all $a \in M$, there exists $e\gamma e = e \in M$ such that $M\Gamma a = M\Gamma e$ and M is left normal. Then $a \in M\Gamma a = M\Gamma e$, so that there exists $y \in M$ such that $a = y\gamma e$ for all $\gamma \in \Gamma$. Then $e = e\gamma e \in M\Gamma e = M\Gamma a$, so that there exists b ∈Msuch that $e = b\gamma a$ for all $\gamma \in \Gamma$. Thus we obtain that $a = y\gamma e = y\gamma_1 e\gamma_2 e = y\gamma_1 e\gamma_2 b\gamma_3 a = a\gamma_1 b\gamma_2 a$ for all $\gamma_1,\gamma_2,\gamma_3 \in \Gamma$. Hence M is regular.

3.4 Theorem:-

Let R be a Γ Seminear ring. Then M is regular if and only if M has the condition for all a ∈M, there exists $e\gamma e = e \in R$ such that $a\Gamma M = e\Gamma M$ and M is right normal.

Proof:-

The proof is similar to the proof of theorem 3.4.

3.5 Definition:-

Let r be a positive integer, we say that M is a left-r-normal (right-r-normal) Γ Seminear ring. If $a \in M\Gamma a^r$ $(a \in a^r\Gamma M)$ for all $a \in M$.

3.6 Proposition:-

Every left-r-normal (right-r-normal) Γ Seminear ring is a left (right) normal Γ Seminear ring.

Proof:-

Let R be a left-r-normal Γ Seminear ring with $r \geq 2$. Clearly, then for all $a \in M$, $a \in M\Gamma a^r = (M\Gamma a^{r-1})\gamma a \subseteq M\Gamma a$, for all $\gamma \in \Gamma$. ThereforeM is a left normal Γ seminear ring.

If M is a right-r-normal Γ Seminear ring with r ≥ 2 for all $a \in M$. We get $a \in a^r\Gamma M = a\gamma(a^{r-1})\Gamma M \subset aM$, for all $\gamma \in \Gamma$. (i.e) $a \in a\Gamma M$. Hence M is a right normal Γ Seminear ring. Hence proved a left and right normal.

3.7 Proposition:-

If M is a left-2-normal Γ Seminear ring, then only homomorphic image M' is also a left-2-normal Γ Seminear ring.

Proof:-

Let g : M → M' be a Γ seminear ring epimorphism $a' \in M'$. Since g is onto. There exists $a \in M$ such that $a' = g(a)$. As M is left-2-normal Γ Seminear ring. There exists b ∈M such that a = $b\gamma_1 a\gamma_2 a$ for all $\gamma_1, \gamma_2 \in \Gamma$. Now, $a' = g(b\gamma_1 a\gamma_2 a) = g(b)\gamma_1 g(a)\gamma_2 g(a) = b'\gamma_1 a'\gamma_2 a'$, Where $b' = g(b) \in M$. Hence M' is left-2-normal.

3.8 Proposition:-

Let M be a left-2-normal Γ Seminear ring. If $b\gamma_1 b = b\gamma_1 b\gamma_2 b$, for some $b \in M$, $\gamma_1, \gamma_2 \in \Gamma$. $b = b\gamma_1 b$.

Proof:-

Since R is a left-2-normal Γ Seminear ring. We can find y ∈M such that $b = y\gamma_1 b\gamma_2 b$, then $b = y\gamma_1 b\gamma_2 b = y\gamma_1(y\gamma_2 b\gamma_3 b)\gamma_4 b = (y\gamma_1 y)\gamma_2(b\gamma_3 b\gamma_4 b) = y\gamma_1 y\gamma_2 b\gamma_3 b$ [since $b\gamma_4 b = b$]. Now, $b = y\gamma_1 y\gamma_2 b\gamma_3 b = y\gamma_1 b$ [since $y\gamma_1 b\gamma_2 b = b$] $b = y\gamma_1 b$. Since $b = y\gamma_1 y\gamma_2 b\gamma_3 b = y\gamma_1(y\gamma_2 b)\gamma_3 b = y\gamma_1 b\gamma_3 b$ [since $b = y\gamma_2 b$] $= b\gamma_3 b$ [since $b = y\gamma_1 b$] =>$b = b\gamma_3 b$. Hence proved

3.9 Proposition:-

If M is a left-2-normal Γ Seminear ring, then every left ideal of M is idempotent.

Proof:-

Let A be a left ideal of M. That implies$M\Gamma A \subseteq A$. Since $A\Gamma A \subseteq M\Gamma A \subseteq A$.

for reverse inclusion $a \in A$, there exists b ∈ R such that $a = b\gamma_1 a\gamma_2 a = M\Gamma A\Gamma A = A\Gamma A$. Hence Proved.

We conclude this paper with the following theorem.

3.10 Theorem:-

Let M be a left – 2 – normal seminear ring. Then

(1) M is reduced.
(2) $a\gamma b = 0$ implies $b\gamma a = 0$.
(3) $a\gamma b = 0$ implies $\langle a\rangle \Gamma \langle b\rangle = 0$ for all $a, b \in R$.

Proof:-

(1) If $a\gamma a = 0$ and $a = b\gamma_1 a\gamma_2 a = b\gamma_1 0$ then $0 = a\gamma a = (b\gamma_1 0)\gamma_2 a = b\gamma_1(0\gamma_2 a) = b\gamma_1 0 = a \Rightarrow a = 0$. Hence R is reduced.

(2) Since $a\gamma b = 0$ we have that $(b\gamma_1 a)\gamma_2(b\gamma_3 a) = b\gamma_1 a\gamma_2 b\gamma_3 a = b\gamma_1(a\gamma_2 b)\gamma_3 a = b\gamma_1 0\gamma_3 a = 0$. Now (1) demands that $b\gamma_1 a = 0$. Hence proved.

(3) Suppose $a\gamma b = 0$. Then $a \in (0 : b)$ and hence $\langle a\rangle \in (0 : b)$ and $\langle a\rangle\Gamma b = \{0\}$. By (2) $b\,\Gamma\langle a\rangle = \{0\}$. Therefore $b \in (0 : \langle a\rangle)$. That is $\langle b\rangle \in (0 : \langle a\rangle)$. Hence $\langle b\rangle$

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