A STUDY ON PSEUDO COMMUTATIVE SEMINEAR RINGS

[1]D. Radha, [2]C. Raja Lakshmi
[1]Assistant Professor of Mathematics, A.P.C. Mahalaxmi College for Women, Thoothukudi, Tamil Nadu, India.
[2]M.Phil Scholar, A.P.C. Mahalaxmi College for Women, Thoothukudi, Tamil Nadu, India.

ABSTRACT:

A near ring N is called pseudo commutative if $xyz = zyx$ for every x, y, z in N. It is quite natural to investigate the corresponding definition in the field seminear ring. In this paper, we define a seminear-ring N to be pseudo commutative seminear-ring if $xyz = zyx$ for every $x, y, z \in N$. It is proved that in a pseudo commutative seminear ring, every idempotent is central and the homomorphic image of pseudo commutative seminear ring is also so. It is observed that every commutative seminear ring is pseudo commutative seminear ring and the converse is true if it has an identity. The condition under which a weak commutative seminear-ring is pseudo commutative and vice versa is obtained. Also we proved that every pseudo commutative seminear-ring has $(*, IFP)$. A pseudo commutative seminear ring M is reduced and also every ideal in M is completely semiprime if it is regular.

KEYWORDS:

Seminear rings, regular, idempotent, central element, commutative seminear rings, weak commutativity, IFP ideals.

I. INTRODUCTION:

The study of semirings was started by the German Mathematician Dedekind [3]. However, during the late 1960s real and significant applications of semirings were found in several field including automata theory, graph theory, coding theory, optimization theory, analysis of computer programs, algebras of formal processes and generalized fuzzy computation. Later on, different people [5, 8, 9] worked semirings and explored many interesting properties in semirings.

In a semi-ring (N, +, .) if we ignore commutativity of + and one distributive law, (N, +, .) is a seminear-ring. If we do not stipulate the left distributive law, (N, +, .) is a right seminear-ring. Wily G Van Hoorn and Van Rootselaar [25] introduced the notion of a seminear-ring which is a generalization of a semiring and near-rings. Especially he discussed homomorphism in seminear-rings and obtained some interesting properties. Further, Balakrishnan [1], Perumal [12, 13] and some others [5, 8, 9, 26] worked in the field of seminear-rings and explored many interesting and elegant properties. Pilz [14] and many other people [2, 7, 11, 24] worked in the field of near-rings and obtained many more properties of near-rings. Dheena [4], Henry [6] introduced regularity in near-rings. In this paper, we make a study on pseudo commutative seminear-rings.

II. PRELIMINARIES:

DEFINITION: II.1 [13]

A *right seminear-ring* is a non-empty set N with two binary operations + and . such that

(i) (N, +) is a semigroup
(ii) (N, .) is a semigroup
(iii) $(x + y)a = xa + ya$ for all $a, x, y \in N$

DEFINITION: II.2 [13, 23, 26]

A right seminear-ring N is said to have an *absorbing zero* if

(i) $a + 0 = 0 + a = a$
(ii) $a. 0 = 0. a = 0$, holds for all $a \in N$.

DEFINITION: II.3 [8, 13]

An element a in the seminear-ring N is said to be *idempotent* if $a^2 = a$

DEFINITION: II.4 [8]

An idempotent element e in the seminear-ring N is called *central* if $ex = xe$ for all $x \in N$.

NOTATION: II.5 [8, 13]

E is the set of all idempotent elements in the seminear-ring N.

C(N) is the set of all central elements in the seminear-ring N.

DEFINITION: II.6 [12]

A seminear-ring N is called *regular seminear-ring* if for each a in N, there exists $x \in N$ such that $a = axa$.

DEFINITION: II.7 [8]

An element a in the seminear-ring N is said to be *nilpotent* if $a^k = 0$ for some least positive integer k.

DEFINITION: II.8

A seminear-ring N is said to be *reduced* if it has no non-zero nilpotent element in N.

DEFINITION: II.9 [13, 14]

A seminear-ring N is said to *fulfil insertion of factor property – IFP* for short – if for $a, b \in N, ab = 0$ implies $anb = 0$ for all n in N.

DEFINITION: II.10

A seminear-ring N is said to have *strong IFP* if $ab \in I$ implies $axb \in I$ for a, b, x in N and I is any ideal in N

DEFINITION: II.11 [13]

A seminear-ring N is said to have $(*, IFP)$ if

(i) N has IFP
(ii) for a, b in N, $ab = 0$ implies $ba = 0$.

DEFINITION: II.12 [14]

An additive group A of the seminear-ring N is called a *N-subgroup* if $NA \subseteq A$ where $NA = \{ra/r \in N, a \in A\}$

DEFINITION: II.13 [8, 23]

A non-empty subset I of a seminear-ring N is called a *left (right) ideal* if,

(i) for all $x, y \in I, x + y \in I$
(ii) for all $x \in I$ and $a \in N$, $ax \in I$ $(xa \in I)$

DEFINITION: II.14 [8, 13, 23]

A non-empty subset I of the semi-near ring N is said to be an *ideal* of N if it is both a left ideal and a right ideal of N.

DEFINITION: II.15 [8, 13]

An ideal I of the seminear-ring N is called a *semi prime ideal* if $A^2 \subseteq I$ implies $A \subseteq I$ holds for all ideals A of N.

DEFINITION: II.16 [8, 13]

An ideal I of the seminear-ring N is called a *completely semi prime ideal* if for x in N, $x^2 \in I$ implies $x \in I$.

DEFINITION: II.17 [13, 26]

If S is any non empty subset of the seminear-ring N, then the *left annihilator of S in N* is $l(S) = \{x \in N / xs = 0\ \forall s \in S\}$

DEFINITION: II.18 [25]

Let $(R,+,\ .)$ and $(S,\ \oplus,\ \odot)$ be two seminear rings. A mapping $\mu : R \longrightarrow S$ is called a *homomorphism* if and only if

(i) $\mu(a+b) = \mu(a) \oplus \mu(b)$
(ii) $\mu(ab) = \mu(a) \odot \mu(b)$

for all a, b in R.

DEFINITION: II.19 [14]

A seminear-ring N has the *property* P_4 if for all ideals I of N, $xy \in I$ implies $yx \in I$ for all x, y in I.

III. PSEUDO COMMUTATIVE SEMINEAR RINGS

Hereafter M stands for Pseudo Commutative Seminear-ring with absorbing zero.

DEFINITION: III.1

A Seminear-ring N is said to be *weak commutative seminear ring* if $xyz = xzy$ for every $x, y, z \in N$.

DEFINITION: III.2

$+$	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	0	0	c

A Seminear ring N is called *pseudo commutative seminear ring* if $xyz = zyx$ for every $x, y, z \in N$.

PROPOSITION: III.3

Let M be pseudo commutative seminear-ring. If E≠ {0}, then E ⊆ C(M).

Proof:

Let e be an idempotent in M. Since M is pseudo commutative seminear-ring and for every $a \in M$, $ae = aee = eae = ea$. This implies $e \in$ C(M).

PROPOSITION: III.4

Homomorphic image of a pseudo commutative seminear-ring is also pseudo commutative.

Proof:

Let M, M' be two seminear-rings and $\varphi: M \longrightarrow M'$ be homomorphism. Let $a', b', c' \in M'$. Then, $a' = \varphi(a), b' = \varphi(b)$ and $c' = \varphi(c)$. Since M is pseudo commutative seminear-ring, $a'b'c' = \varphi(a)\varphi(b)\varphi(c) = \varphi(ab)\varphi(c) = \varphi(abc) = \varphi(cba) = \varphi(cb)\varphi(a) = \varphi(c)\varphi(b)\varphi(a) = c'b'a'$. Hence the proof.

PROPOSITION: III.5

Every commutative seminear-ring is pseudo commutative.

Proof:

Let N be a commutative seminear-ring and M be a pseudo commutative seminear-ring. Let $x, y, z \in N$. Since N is a commutative seminear-ring, $xyz = x(yz) = x(zy) = (xz)y = z(xy) = z(yx) = zyx$ for all x, y, z in N.

EXAMPLE: III.6

Consider the seminear-ring $(N,+,\ .)$ where $N = \{0, a, b, c\}$ with $(N,+)$ and $(N,.)$ is given as

Then N is a pseudo commutative seminear-ring but not commutative since $bc \neq cb$.

EXAMPLE: III.7

The seminear-ring $(N, +, .)$ where $N = \{0, a, b, c\}$ with $(N, +)$ defined as in example III.6 and $(N, .)$ defined by

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	0	0

is a pseudo commutative seminear-ring and is commutative.

PROPOSITION: III.8

Any pseudo commutative seminear-ring M with identity is commutative.

Proof:

Let $a, b, c \in M$ and u be the identity of M. Since M is pseudo commutative seminear-ring and for every a, b, c in M, $ab = abu = uba = ba$. This implies M is commutative.

PROPOSITION: III.9

Any weak commutative seminear-ring with left identity is a pseudo commutative seminear-ring.

Proof:

Let N be weak commutative seminear-ring. Then for each a, b, c in N, $abc = (ea)bc = e(acb) = e(acb) = (eac)b = (eca)b = e(cab) = e(cba) = cba$. This implies N is a pseudo commutative seminear-ring.

PROPOSITION: III.10

Any pseudo commutative seminear-ring M with right identity is weak commutative seminear-ring.

Proof:

Let a, b, c be in M and e be the right identity. Then for each a, b, c in M, $abc = abce = a(bce) = a(ecb) = (ae)(cb) = acb$. Thus M is weak commutative seminear-ring.

THEOREM: III.11

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	0	0
c	0	c	0	0

Let M be a regular pseudo commutative seminear-ring. Then M is reduced

Proof:

Since M is regular, for every a in M, there exists b in M such that $a = aba$. Then $(ab)^2 = (ab)(ab) = (aba)b = ab$. Thus $ab \in E\ \forall\ a, b \in M$. By proposition III.3, $a = aba = (ab)a = a(ab) = a^2b$. If $a^2 = 0$, then $a = 0.b = 0$. This implies M is reduced.

THEOREM: III.12

Let M be a pseudo commutative seminear-ring. Then M has (, IFP)*

Proof:

Let $ab = 0$ for all $a, b \in M$. Then $(ba)^2 = (ba)(ba) = b(ab)a = b0a = b0 = 0$. By theorem III.11, $ba = 0$. And for every n in M, $(anb)^2 = (anb)(anb) = an(ba)nb = an0nb = 0$. Again by theorem III.11, $anb = 0$. This implies M has (*, IFP).

LEMMA: III.13

Let N be regular seminear-ring. Then, $axN = aN$ and $Nxa = Na$ for $a \in N$.

Proof:

Since N is regular, for each a in N, there exists $x \in N$ such that $a = axa$. Let

$y \in axN$. Thus, $y = axb \in aN$ for b in N. Let $z \in aN$. Then, $z = ax = axax \in axN$. This completes the lemma.

LEMMA: III.14

A seminear-ring N has IFP if and only if $(0{:}S)$ is an ideal for any subset S of N.

Proof:

Let $I = l(S) = \{x \in N / xs = 0\}$. For $s \in S, x, y \in I, (x+y)s = xs + ys = 0$ implies that $x + y \in I$. Further $x \in I, y \in N$ we have $yxs = y0 = 0$ which implies $yx \in I$. And let $x \in I$. Then, $xs = 0$. Since N has IFP we have $xys = 0$ for all y in N. This implies $xy \in I$. Therefore, I is an ideal of N.

THEOREM: III.15

Let M be a regular pseudo commutative seminear-ring. Then every N-subgroup is an ideal.

Proof:

Let $a \in$ M. Since M is regular, $a = aba$ for some $b \in M$. And $(ba)^2 = (ba)(ba) = b(aba) = ba$. This implies ba is an idempotent element in M. Let $ba = e$. By lemma: III.12, $Me = Mba = Ma$. Let $S = \{m - me / m \in M\}$. Now, we show that (0:S) = Me. Since, $(m - me)e = me - mee = me - me = 0 \; \forall \, m \in M$. This implies $(m - me)Me = 0$. Thus, $Me \subseteq (0{:}S)$. Let $y \in (0{:}S)$. Then $y = yxy$ for some x in M and $yx - (yx)e \in S$. Therefore, $(yx - yxe)y = 0$ and this implies $yxy - yxey = 0$. Thus we get, $0 = y - y(xey) = y - y(xye) = y - (yxy)e = y - ye$. Hence, $(0{:}S) \subseteq Me$. Therefore, $(0{:}S) = Me = Ma$. By lemma: III.14, Ma is an ideal of M. Now if P is any N-subgroup of M then, P = $\sum_{a \in P} Na$. Thus, P becomes an ideal of M.

THEOREM: III.16

Let M be regular pseudo commutative seminear-ring. Then $M = Ma = Ma^2 = aM = aMa$ for all a in M.

Proof:

Since M is regular, for every a in M, there exists b in M such that $a = aba = a(ba) = (ba)a = ba^2 \in Ma^2$. Therefore, $M \subseteq Ma^2$. Now, $Ma \subseteq M \subseteq Ma^2 \subseteq Ma \subseteq M$ implies, $Ma = Ma^2 = M$. Now we claim that $Ma^2 = aM$. Let $x \in Ma^2$. Then by the definition of pseudo commutative seminear-ring, for some m in M, $x = ma^2 = maa = ama \in aM$. Therefore, $Ma^2 \subseteq aM$. Let $am \in aM$. Since M is regular pseudo commutative seminear-ring, $am = (aba)m = a(ba)m = (ba)am = b(aam) = b(maa) = bma^2 \in Ma^2$. This implies, $Ma^2 = aM$. Now we claim that $aM = aMa$. Since M is regular and Ma is an ideal, for every m in M, $am = (aba)m = a(bam) \in a(MaM) \subseteq aMa$. Thus, $aM \subseteq aMa$. Obviously, $aMa \subseteq aM$. Hence $aM = aMa$ for all a in M.

THEOREM: III.17

Let M be a regular pseudo commutative seminear-ring. Then any ideal of M is completely semi prime.

Proof:

Let $a^2 \in I$. Then, $a = aba = (ab)a = a(ab) = a^2 b \in IM \subseteq I$. Thus, every ideal of M is completely semi prime.

LEMMA: III.18

Every ideal I of the seminear-ring N satisfies $NIN \subseteq I$.

Proof:

Let $x \in NIN$. Thus $x = aib \in Ib \in I$. This gives $NIN \subseteq I$

THEOREM: III.19

Let M be regular pseudo commutative seminear-ring. Then M has property P_4

Proof:

Let $ab \in I$. Then by lemma: III.18, $(ba)^2 = ba(ba) = b(ab)a \in MIM \subseteq I$. By theorem: III.17, $ba \in I$. Hence the theorem.

THEOREM: III.20

Let M be a regular pseudo commutative seminear-ring. Then M has strong IFP.

Proof:

Let I be an ideal of M. Since $MI \subseteq I$, by theorem: III.16, we have $aM = Ma^2$. Hence $an = ma^2$ for some $n, m \in M$. Hence if $ab \in I$, then for every n in M, $anb = ma^2b = (ma)(ab) \in MI \subseteq I$. Therefore, M has strong IFP.

REFERENCES:

[1] Balakrishnan R and Perumal R *Left duo seminear-rings*, Scientia Magna 2012, 8(3), 115-120.

[2] Booth G L and Groenewald N J *On Strongly prime near-rings*, Indian Journal of Mathematics 1998, 40(2), 113-121

[3] Dedekind R *Uber die Theorie der ganzenalgebraiscen zahlen*, Springer Fachmedian 1964,297-313.

[4] Dheena P, *On Strongly regular near-rings*, Journal of the Indian Math. Soc., 1985(49), 201-208.

[5] Golan J S *Semirings and their applications*, Kluwar Academic publishers, 1999.

[6] Henry E. Heartherly, *Regular Near-rings*, Journal of Indian Math. Soc., 1974(38), 345-354.

[7] Jat J L and Choudary S C *On Left Bipotent Near-rings*, Edwin. Math. Soc., 1979(22), 99-107.

[8] Javed Ahsan, *Seminear-rings characterized by their s-ideals –I*, Proc. Japan Acad., 1995, 71(A), 101-103.

[9] Javed Ahsan, *Seminear-rings characterized by their s-ideals–II*, Proc. Japan Acad., 1995, 71(A), 111-113.

[10] Oswald A, *Near-rings in which every N-subgroup is principal*, Proc. London Math. Soc., 1974, 28(3), 67-68.

[11] Park Y S and Kim W J, *On structures of left bipotent Near-rings*, Kyungbook Math. J, 1980, 20(2), 177-181.

[12] Perumal R and Balakrishnan R, *Left Bipotent seminear-rings*, International Journal of Algebra, 2012, 6(26), 1289-95.

[13] Perumal R, Arulprakasam R and Radhakrishnan M, *A note on ideals in Seminear-rings*, IOP conf. Series: Journal of Physics: conf. Series 1000(2018)012150.

[14] Pilz Gunter, *Near-rings*, North Holland, Amsterdam, 1983.

[15] Radha D and Kavitha M, *Pseudo Symmetric Ternary Γ - Semiring*, Proceedings on National Conference on Innovations in Mathematics (NCIM - 2018), Feburary 2018, ISBN : 978-81-935198-5-1, Page (19-24).

[16] Radha D and Meenakshi P, *Some Structures of Idempotent Commutative Semigroup*, International Journal of Science, Engineering and Management

(IJSEM) Vol 2, Issue 12, December 2017, ISSN (Online) 2456-1304.

[17] Radha D and Parvathi Banu M, Left Singularity and Left Regularity in Near Idempotent Γ - Semigroup , International Journal of Science, Engineering and Management (IJSEM) Vol 3, Issue 4, April 2018, ISSN (Online) 2456-1304.

[18] Radha D, Raja Lakshmi C, *On Weakly π-subcommutative* Γ-*Near Rings*, Proceedings; National Conference on Innovations in Mathematics(NCIM-2018) Feb 9, 2018, (10-18), ISBN No.:978-81-935198-5-1.

[19] Radha D, Raja Lakshmi C, *A study on Semicentral Seminear-rings*, Proceedings; National Seminar on New Dimensions in Mathematics and its Applications, October 17, 2018, ISBN No.:(yet to receive).

[20] Raja Lakshmi C, Radha D, *On zero-symmetric Semicentral* Γ-*Near Rings*, International Journal of Science, Engineering and Management (IJSEM) vol 3, Issue 4, April 2018, 571-575, ISSN (online) 2456-1304

[21] Radha D and Selvi V, *Stable and Pseudo Stable Gamma Near Rings*, Proceedings on National Conference on Recent Trends in Pure and Applied Mathematics, September 2017, ISBN: 978-81-935198-1-3.

[22] Radha D and Suguna S, *Normality in Idempotent Commutative Γ -Semigroup*, International Journal of Science, Engineering and Management (IJSEM) Vol 3, Issue 4, April 2018, ISSN (Online) 2456-1304.

[23] Shabir M and Ahamed I, *Weakly regular Seminear-rings*, International Electronic Journal of Algebra, 2007, 114-126.

[24] Suryanarayana S and Ganesan N, *Stable and Pseudo Stable Near-rings*, Indian Journal of Pure and Applied Math, 1988, 19(12), 1206-16.

[25] Wily G Van Hoorn and Van Rootselaar B, *Fundamental notions in the theory of Seminear-rings*, Compositio Mathematic, 1976, 18, 65-78

[26] Weinert H J, *Seminear-rings, Seminear-field and their semigroup theoretical background*, Semigroup Forum, 1982, 24, 231-54.