AN INNOVATIVE APPROACH TO SOLVE NONLINEAR INTEGRO DIFFERENTIAL EQUATION: A CUBIC LEGENDRE SPLINE COLLOCATION METHOD

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Abstract. This paper is an attempt to solve the nonlinear integro differential equation by an innovative approach known as Legendre spline collocation method. This innovation gives better result compare to previous traditional methods to solve a nonlinear integro differential equation. This method reduces complicated nonlinear integro differential equation into the simple nonlinear system of equations. The accurate result has been obtained using this method. The derived results justify applicability and accuracy of this newly developed method.

Keywords: **Legendre polynomial, spline collocation, Integro differential equation.**

1 Introduction

It has been observed that many mathematical models for solving science and engineering problems are in the form of linear or nonlinear Integro differential equation. Most of the real-world problems are nonlinear whose analytical solution is impractical. So, it is indispensable to find approximate solution of nonlinear Integro differential equation.

Many attempts have been done to solve nonlinear Integro differential equations before, like Block by Block and Finite Difference Hybrid Method [2], Tau method [3], Hybrid of Block-Pulse Functions and Lagrange Interpolating Polynomials [4], power series method [5], haar wavelets [6], Bernstein Polynomials [7]

Most important numerical approximation is polynomial approximation. Spline polynomial approximation is in vogue to solve higher mathematical problems. In past, number of higher mathematical problems like Differential equation, Integral equation, Partial differential equation, and fractional Differential equation were solved by using spline function.

There is not a well expressed orthogonal basis in spline space to date [8]. This paper represents a new orthogonal spline basis with the help of Legendre polynomial, which has been used to solve the first-order Integro differential equations of the form:

$$y'(x) = f(x) + \int_a^x k_1(x,t,y(t))dt + \int_a^b k_2(x,t,y(t))dt, \quad a \le x \le b$$

With

$$y(a) = y(x_0) = y_0$$

where the functions f(x), g(x) and k (x, s) are sufficiently smooth real-valued functions.

The remainder of this paper is organized as follows: In section 2, we describe the linearization method to convert nonlinear integro differential equation to linear integro differential equation. In Section 3, Description of converting nonlinear VIDE to linear VIDE has been discussed. In section 4, The basic formulation of Legendre spline polynomial required to solve nonlinear VIDE has been given. Section 5 is devoted to the applicability and accuracy of the proposed method by considering numerical examples. Also, a conclusion is given in the last Section.

2 Solution Of Nonlinear Integro Differential Equation

2.1 Linearization Method

Let us consider the following nonlinear Volterra-Fredholm integro differential equation

$$y'(x) = f(x) + \int_a^x k_1(x,t,y(t))dt + \int_a^b k_2(x,t,y(t))dt, \quad a \le x \le b$$

where y(x) is the unknown function, a and b are constants and the functions f(x), $k_1(x,t,y)$ and $k_2(x,t,y)$ are analytical on R and R3, respectively. Here $k_i(x,t,y(t))$ is a nonlinear function and which may be singular at $x = x_j$.

Also, initial condition is to be assumed as

$$y(a) = y(x_0) = y_0.$$

Let us find now solution of above equation using given initial condition by dividing the given interval [a, b] into a series of subintervals $I_j = [x_j, x_j+1]$ such that $x_0 = a$. In each sub interval $k_i\big(x,t,y(t)\big)$, is linearized by taking first three terms of the expansion of it using Taylor series around the point (x_i, y_i, z_i), in the following form.

$$k_i(x,t,y) = k_{ij} + J_{ij}(x-x_j) + T_{ij}(t-t_j) + P_{ij}(y-y_j), \quad i = 1,2$$

Where

$$k_{ij} = k_i(x_j,t_j,y_j),\ J_{ij} = \frac{\partial k_i(x_j,t_j,y_j)}{\partial x}, T_{ij} = \frac{\partial k_i(x_j,t_j,y_j)}{\partial t}, P_{ij} = \frac{\partial k_i(x_j,t_j,y_j)}{\partial y}, \quad i = 1,2$$

substituting $k_i\big(x,t,y(t)\big)$ in Volterra part of given equation, we get:

$$\int_a^x k_1\big(x,t,y(t)\big)dt$$

$$\cong \int_a^x k_{1j} + J_{1j}(x-x_j) + T_{1j}(t-t_j) + P_{1j}(y-y_j)dt$$
$$= [k_{1j} + J_{1j}(x-x_j) - P_{1j}y_j](x-x_0) + \frac{T_{1j}}{2}[(x-t_j)^2 - (x_0-t_j)^2] + P_{1j}\int_a^x y(t)\,dt$$

substituting $k_i\big(x,t,y(t)\big)$in Fredholm part of given equation, we get:

$$\int_a^b k_2\big(x,t,y(t)\big)dt$$

$$\cong \int_a^b k_{2j} + J_{2j}(x-x_j) + T_{2j}(t-t_j) + P_{2j}(y-y_j)dt$$

$$= [k_{2j} + J_{2j}(x-x_j) - P_{2j}y_j](b-a) + \frac{T_{2j}}{2}[(b-a)(b+a-2t_j)] + P_{2j}\int_a^b y(t)dt$$

Using above formulation, we will convert nonlinear equation into linear equation.

3 To convert the nonlinear VDIE to linear VIDE

Let us convert the following nonlinear volterra integro differential equation in to linear volterra integro differential equation.

$$y'(x) - y(x) + \int_0^x k\big(x,t,y(t)\big)dt = f(x)$$

Substitute $k_i(x_i,t,y) = k_{ij} + J_{ij}(x-x_j) + T_{ij}(t-t_j) + P_{ij}(y-y_j)$

So, above equation becomes

$$y'(x) - y(x) + \sum_{j=0}^{i-1}\int_{x_j}^{x_{j+1}} [k_{ij} + J_{ij}(x-x_j) + T_{ij}(t-t_j) + P_{ij}(y-y_j)]dt = f(x)$$

Now using,

$$\int_a^x k_1\big(x,t,y(t)\big)dt$$

$$\cong \int_a^x k_{1j} + J_{1j}(x-x_j) + T_{1j}(t-t_j) + P_{1j}(y-y_j)dt$$
$$= [k_{1j} + J_{1j}(x-x_j) - P_{1j}y_j](x-x_0) + \frac{T_{1j}}{2}[(x-t_j)^2 - (x_0-t_j)^2] + P_{1j}\int_a^x y(t)\,dt$$

We get,

$$y'(x)-y(x)+\sum_{j=0}^{i-1}[k_{1j}+J_{1j}(x-x_j)-P_{1j}y_j](x_{j+1}-x_j)$$
$$+\frac{T_{1j}}{2}[(x_{j+1}-t_j)^2-(x_j-t_j)^2]+P_{1j}\int_{x_j}^{x_{j+1}}y_j(t)dt=f(x)$$

Collocating above equation at x = x_i for i = 0, 1 ..., n.

$$y'(x_i)-y(x_i)+\sum_{j=0}^{i-1}[k_{1j}+J_{1j}(x_i-x_j)-P_{1j}y_j](x_{j+1}-x_j)$$
$$+\frac{T_{1j}}{2}\left[(x_{j+1}-t_j)^2-(x_j-t_j)^2\right]+P_{1j}\int_{x_j}^{x_{j+1}}y_j(t)dt$$
$$=f(x_i) \qquad (1)$$

4 Legendre Spline Polynomial

Let u(x) be a function defined on an interval [a, b] and $\Delta: a = x_0 < x_1 < \cdots \ldots\ldots\ldots\ldots < x_N = b$, be a given partition of the interval [a, b]. Each point x_i (i = 0 (1) N) is also called a knot or node, for the space of spline functions.

A polynomial function $s(x) \in C^2[a,b]$, which interpolates u(x) at the mesh points x_i,

$i=0(1)N$ is called a Legendre cubic spline, if it coincides with a Legendre cubic polynomial $s_i(x)$ in each subinterval $[x_i, x_{i+1}]$. Let u_i approximate $u(x_i)$ obtained by a segment $s_i(x)$ passing through points (x_i, u_i) and (x_{i+1}, u_{i+1})

For each segment $[x_i, x_{i+1}], s_i(x)$ has the following form

$$s_i(x)=\frac{1}{2}a_i[5(x-x_i)^3-3(x-x_i)]+\frac{1}{2}b_i[3(x-x_i)^2-1]+c_i(x-x_i)$$
$$+d_i \qquad (2)$$

where, $s_i(x_i)=y_i,\ s_i(x_{i+1})=y_{i+1}, s_i''(x_i)=M_i, s_i''(x_{i+1})=M_{i+1}$ and $h=\frac{b-a}{N}$ for i = 0 (1) N

$$a_i=\frac{M_{i+1}-M_i}{15h}$$
$$b_i=\frac{M_i}{3}$$
$$c_i=\frac{1}{h}\left[(y_{i+1}-y_i)-M_{i+1}\left(\frac{h^2}{6}-\frac{1}{10}\right)-M_i\left(\frac{h^2}{3}+\frac{1}{10}\right)\right]$$
$$d_i=y_i+\frac{M_i}{6}$$

So that equation (2) can now be written as

$$s_i(x)=\frac{M_{i+1}-M_i}{6h}(x-x_i)^3+\frac{M_i}{2}(x-x_i)^2$$
$$+\left[\frac{1}{h}(y_{i+1}-y_i)-\frac{h}{6}M_{i+1}-\frac{h}{3}M_i\right](x-x_i)+y_i$$
$$(3)$$

Which is known as Cubic Legendre Spline polynomial and using it unknown y(x) can be determined.

5 Examples

5.1 Consider the following nonlinear IDE [G. Ebadi et al (2007) [2]]

$$y'(x)-y(x)+2\int_0^x xte^{-y^2(t)}\,dt=1-xe^{-x^2};\ x\in[0,1],\quad y(0)=0$$

Solution:

Here $k(x,t,y(t))=xte^{-y^2(t)}$

so,

$$k(x_i,t_j,y_j)=x_it_je^{-y^2(t_j)}$$
$$J_{1j}=\frac{\partial k}{\partial x}=t_je^{-y_j^2},$$
$$T_{1j}=\frac{\partial k}{\partial t}=x_je^{-y_j^2},$$
$$P_{1j}=\frac{\partial k}{\partial y}=-2x_jt_je^{-y_j^2}y_j$$

Substitute all in given equation and using equation (1)

We get,

$$y'(x_i) - y(x_i) + 2\sum_{j=0}^{i-1}\left[x_j t_j e^{-y_j^2} + t_j e^{-y_j^2}(x_i - x_j) + 2x_j t_j e^{-y_j^2} y_j\right](x_i - x_j) + \frac{x_j e^{-y_j^2}}{2}\left[(x_{j+1} - t_j)^2 - (x_j - t_j)^2 - 2x_j t_j e^{-y_j^2} y_j \int_{x_j}^{x_{j+1}} y_j(t)\,dt\right] = 1 - x_i e^{-x_i^2}$$

For n = 2, i = 0, 1, 2 and approximating unknown y(x) by using cubic Legendre spline polynomial (3), we get

$$-\frac{M_0}{6} - \frac{M_1}{12} - 3y_0 + 2y_1 = 1$$

$$\frac{M_0}{12} + \frac{M_1}{6} - 2y_0 + y_1 = 0.6106$$

$$\frac{M_1}{12} + \frac{M_2}{6} - 2y_1 + y_2 + \left[\frac{3}{8}{y_1}^2 + \frac{M_2 y_1}{384} + \frac{M_1 y_1}{384} - \frac{y_1 y_2}{8}\right] e^{-{y_1}^2} = 0.6321$$

Taking $M_0 = 0,\ M_1 = y_3,\ M_2 = y_4$ and using initial condition and continuity condition

$$M_{i+1} + 4M_i + M_{i-1} = \frac{6}{h^2}[y_{i+1} - 2y_i + y_{i-1}],\ \ i = 1, 2, \dots\dots, n-1$$

we get, the following nonlinear system of equations, which can be than solved by newton Raphson method.

$$-\frac{y_3}{12} + 2y_1 = 1$$

$$-\frac{y_3}{6} + y_1 = 0.6106$$

$$\frac{y_4}{6} + \frac{y_3}{12} - 2y_1 + y_2 + \left[\frac{9}{16} + \frac{3{y_1}^2}{8} + \frac{y_4 y_1}{384} + \frac{y_3 y_1}{384} - \frac{y_1 y_2}{8}\right] e^{-{y_1}^2} = 0.6321$$

$$4y_3 + y_4 + 48y_1 - 24y_2 = 0$$

By solving these equations, we get

$y_1 = 0.4631,\quad y_2 = 0.9265,\quad y_3 = -0.8848,\quad y_4 = -1.2557$

Table 1. CLSC solution of nonlinear IDE 5.1

x:	CLSC Solution:	Tau Method:	Exact Solution: x
0	0	0	0
0.1	0.0999	0.100	0.100
0.2	0.1994	0.2	0.2
0.3	0.2980	0.3	0.3
0.4	0.3953	0.4	0.4
0.5	0.4631	0.5	0.5
0.6	0.5765	0.6	0.6
0.7	0.6809	0.7	0.7
0.8	0.7761	0.8	0.8
0.9	0.8619	0.9	0.9
1.0	0.9381	1	1

5.2. Consider the following nonlinear IDE [yadollah ordokhani (2011) [6]]

$$y'(x) = 1 - \frac{x}{3} + \int_0^1 xy^2(t)\,dt;\quad 0 \le x < 1;\quad y(0) = 0$$

Solution:

Here

$$k(x, t, y(t)) = xy^2(t)$$

so,

$$k(x_i, t_j, y_j) = x_j {y_j}^2(t)$$

$$P_{1j} = \frac{\partial k}{\partial y} = 2x_j y_j$$

$$J_{1j} = \frac{\partial k}{\partial x} = {y_j}^2(t)$$

$$T_{1j} = \frac{\partial k}{\partial t} = 0$$

Following the steps of example (1) we can get the following nonlinear system of equations, which than solved by Newton Raphson method.

$$-\frac{y_3}{12} + 2y_1 = 1$$

$$\frac{y_3}{6} + 2y_1 + \frac{{y_1}^2}{4}\left[\frac{y_4 y_1}{192} + \frac{y_3 y_1}{192} - \frac{y_1 y_2}{4}\right] = \frac{5}{3}$$

$$\frac{y_4}{6} + \frac{y_3}{12} - 2y_1 + 2y_2 - \frac{{y_1}^2}{4}\left[\frac{y_4 y_1}{192} + \frac{y_3 y_1}{192} - \frac{y_1 y_2}{4}\right] = \frac{2}{3}$$

$$4y_3 + y_4 + 48y_1 - 24y_2 = 0$$

By solving it, we get

$$y_1 = 0.5321, \quad y_2 = 1.0272, \quad y_3 = 3.1706, \quad y_4 = -3.3715$$

Table 2. CLSC solution of nonlinear IDE 2

x:	CLSC Solution:	Exact Solution: x
0	0	0
0.1	0.0803	0.1
0.2	0.1621	0.2
0.3	0.2471	0.3
0.4	0.3369	0.4
0.5	0.5321	0.5
0.6	0.6217	0.6
0.7	0.7300	0.7
0.8	0.8829	0.8
0.9	1.00	0.9
1.0	1.02	1.0

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