A Modelling Approach to Study the Effect of Delay on Plant Growth Under the Effect of Toxic Metal

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Abstract: In this paper, a mathematical model is proposed to study the effect of delay on plant growth under the effect oftoxic metal. The model is developed using a non-linear system of delay differential equations. In the model, there are four state variables: concentration of toxic metal in soil, density of favorable resource in the plant environment, density of plant biomass and concentration of nutrient in soil. It is assumed the presence of toxic metal in the soil hinders nutrient uptake and nutrient utilization.It further delays the conversion of resources into plant biomassand hence, affects the plant growth adversely. This entire scenario is studied by introducing delay parameter in state variable: favorable resources. Stability of interior equilibrium points is studied whereHopf bifurcation is seen andsensitivity analysis is done.Model is also verified using already existing experimental data for growth of Green vegetables (Brassica juncea L. Czern) under the effect of toxic metals (Cu N.X.2015). Graphical support is provided to analytical results using MATLAB.

Key Words: Nutrients,Plant biomass, Interior equilibrium point, Time delay, Stability, Hopf bifurcation.

Introduction

A minimum level of pH value is essential for survival and growth of plants. But accumulation of acid over a long period of time lowers this pH level and leads to increased level of toxicity of metals. This lower value of pH makes the conditions unsuitable for survival of plants. Weathering of soil, rocks, drainage and mining introduces metal in plant soil dynamics. The solubility of metals is very high in acid solution and thereby increases toxicity of theses metals which adversely affects the plant growth and uptake rate of nutrients. Mathematical modelling was first applied by Thornley [16]to plant physiology where he modeled each process of plant soil dynamics with the help of ordinary differential equations. But, Lacointe [12] studied the models given by [16] and concluded that these are the plant specific models and cannot be applied to all plants in general. De Leo et al. [5]showed with a simple mathematical model that how the combination of soil chemistry and toxic metal can be so adverse for tree biomass. With modifications in the parameters of the modelgiven by De Leo. Guala et al. [6,7]came up with a model that can be applied to all kinds of plant population in general. Ruan and Wei [15] discussed the distribution of roots of exponential polynomials for study of stability involving delays with the help of Rouches theorem. Naresh et al.[13] gave a nonlinear mathematical model to study the effect of delay on plant biomass due to toxicants releases in atmosphere from different sources.It showed how excessive release of toxicants leads to decrease in plant biomass equilibrium level. Huang et al. [8] analyzed global stability of population growth using system of non-linear delay differential equations. Chaturvedi et al.[3] studied the effect of pollutant and toxicants on fish population.Bocharov and Rihan [2] gave adjoint and direct methods for sensitivity analysis in numerical modelling in biosciences using delay differential equations. Rihan [14] did the Sensitivity analysis for dynamic systems with time-lags using adjoint equations and direct methods when the parameters appearing in the model are not only constants but also variables of time. Banks et al. [1] presented theoretical foundations for traditional sensitivity and generalized sensitivity functions for a general class of nonlinear delay differential equations. They Included theoretical results for sensitivity with respect to the delays. Ingalls et al. [9] developed A parametric sensitivity analysis for periodic solutions of delay differential equations.Cu [4] studied the effect of heavy metals on plant growth of Brassica Juncea L. Czern. Kalra and Kumar [10] studied the role of time lag in plant growth dynamics using a two-compartment mathematical model.Kalra and Kumar [11] studied the plant biomass with delay under the effect of toxic metal in the soil and within the plant itself. In recent times, we have not found much of the use of delay differential equationsin study of plant growth and agriculture production. With conception of this idea, a mathematical model is developed to study the plant growth by introducing delay parameter in the terms having consumption and utilization coefficient of resources by plant biomass.

Mathematical Model

Let the four state variables be: Density of favorable resources in plant environment R, Density of plant biomass B, Concentration of nutrients in soil N and Concentration of toxic metalin soil T.This dynamic is governed by following system of non-linear delay differential equations:

$$\frac{dR}{dt} = a_1NR - a_2R - b_1BR(t-\tau) \qquad (1)$$

$$\frac{dB}{dt} = b_1BR(t-\tau) - b_2B - KB^2 \qquad (2)$$

$$\frac{dN}{dt} = N_0 - c_1N - a_1NR - d_2TN + Kb_2B + Ka_2R + KB^2 \qquad (3)$$

$$\frac{dT}{dt} = T_0 - d_1T - d_2TN \qquad (4)$$

With initial conditions $R(0) > 0, B(0) > 0, N(0) > 0, T(0) > 0$ for all $t > 0$ and $R(t-\tau) =$Constant for $t \in [-\tau, 0]$.

The considered parameters are defined as:$T_0$is the constant input rate of pollutant (acid and toxic metal) in soil, $N_0$is constant nutrient availability in soil, $c_1$is the nutrient leaching rate, $a_1$is interactionrate between nutrient and resource, $d_2$is the rate of uptake of toxic metal, $b_1$is the specific rate of utilization of resources by plant biomass, $a_2$is the natural death rate of resource, $b_2$is the natural decay rate of plant biomass, $d_1$is the natural decay rate of pollutant in the soil, K $(0 < k < 1)$ determines the proportionate amount of resource and biomass that is being recycled to nutrient pool after decay. Here $T_0, N_0, c_1, a_1, d_2, b_1, a_2, b_2, d_1, K$are positive constants.

Boundedness

The following lemma shows that solutions of the model (1) – (4) are bounded:

Lemma1. All the solutions of the model (1) – (4) lie in the region: $\Omega = \left\{(R,B,N,T) \in R_+^4, 0 \le \frac{N_0+T_0}{\theta_1} \le (R+B+N+T); 0 \le (R+B+N) \le \frac{N_0}{\theta}; 0 \le T \le \frac{T_0}{d_1}\right\}$as $t \to \infty$ for all positive initial values $\big(R(0), B(0), N(0), T(0)\big) \in R_+^4$and $R(t-\tau) =$Constant for $t \in [-\tau, 0]$where $\theta = min\big((1-K)a_2, (1-K)b_2, c_1\big)$ and $\theta_1 = max\left(\left(c_1 + 2d_2\frac{T_0}{d_1}\right), a_2, b_2, c_1\right)$.

Proof: Let $F(t) = R(t) + B(t) + N(t) \Longrightarrow \frac{dF}{dt} = \frac{d}{dt}(R+B+N)$

Let $\theta = min\big((1-K)a_2, (1-K)b_2, c_1\big)$, we get $\frac{dF}{dt} \le N_0 - \theta F$

By application of usual comparison theorem, we get as $t \to \infty, F \le \frac{N_0}{\theta}$

So, $(R+B+N) \le \frac{N_0}{\theta}$ and from equation (4), we get$T \le \frac{T_0}{d_1}$

Again, let $F_1(t) = F(t) + T(t)$. If $\theta_1 = max\left(\left(c_1 + 2d_2\frac{T_0}{d_1}\right), a_2, b_2, c_1\right)$,

Then $F_1 \ge N_0 + T_0 - \theta_1F_1$

By application of usual comparison theorem, we get as $t \to \infty, F_1 \geq \frac{N_0+T_0}{\theta_1}$

So, $(R+B+N+T) \geq \frac{N_0+T_0}{\theta_1}$. Hence, lemma is proved.

Positivity of Solutions

Positivity of the solutions means the system sustains. It can be done by showing that all solution of system given by Equations. (1)– (4), where initial condition is $R(0) > 0, B(0) > 0, N(0) > 0, T(0) > 0$ for all t and $R(t-\tau) =$ constant for $t \in [-\tau, 0]$, the solution $(R(t), B(t), N(t), T(t)$)of the model stays positive for all time $t > 0$.

From equation (4), $\frac{dT}{dt} \geq -(d_1 + d_2 N)T \Longrightarrow \frac{dT}{dt} \geq -\left(d_1 + d_2 \frac{N_0}{\theta}\right)T$

$$\Longrightarrow \frac{dT}{T} \geq -\left(d_1 + d_2 \frac{N_0}{\theta}\right)dt \Longrightarrow T \geq e^{-\left(d_1+d_2\frac{N_0}{\theta}\right)t}$$

Same argument holds for R, B, N.

Uniform and Interior Equilibriums of the Model

Here we will study three equilibriums $E_i (i = 1,2,3)$ possessed by the model (1) – (3), out of which $E_1$and $E_2$are two non-zero uniform equilibriums and $E_3$is the non-zero feasible interior equilibrium. Here, $R(t-\tau) = R(t)$ for all the points of equilibriums $E_i (i = 1,2,3)$.

(I) The 1st uniform equilibrium $E_1(\bar{N} \neq 0, \bar{R} = 0, \bar{B} = 0, \bar{T} \neq 0)$:

Here $\bar{N} = \frac{N_0}{c_1 + d_1 T}$, and $\bar{T} = \frac{-g_2 + \sqrt{g_2^2 - 4g_1 g_3}}{2g_1} > 0$ if $\sqrt{g_2^2 - 4g_1 g_3} > g_2, -4g_1 g_3 > 0$

Where $g_1 = d_1, g_2 = c_1 - d_1^2 T_0, g_3 = -(c_1 d_1 T_0 + d_2 N_0)$

One root (out of four roots) given by characteristic equation corresponding to equilibrium point E_1 is $\mu = a_1 \bar{N} - a_2$. So, by Routh-Hurwitz criteria, E_1 is locally asymptotically stable if $\bar{N} < \frac{a_2}{a_1}$ which implies that equilibrium level of nutrient concentration in soil is less than fraction of natural death rate of resource to the rate of consumption of nutrient by resource. Further it is observed that E_3 exist only when E_1 is unstable that is $\bar{N} > \frac{a_2}{a_1}$and in this case both resource and plant biomass will die out.

(II) The 2nd uniform equilibrium$E_2\left(\bar{\bar{N}} \neq 0, \bar{\bar{R}} \neq 0, \bar{\bar{B}} = 0, \bar{\bar{T}} \neq 0\right)$:

Here $\bar{\bar{N}} = \frac{a_2}{a_1}, \bar{\bar{T}} = \frac{T_0}{a_1 d_1 + a_2 d_2}, \bar{\bar{R}} = \frac{a_1 N_0 (a_1 d_1 + a_2 d_2) - [a_2 c_1 (a_1 d_1 + a_2 d_2) + a_1^2 d_1 T_0]}{(1-K) a_1 a_2 (a_1 d_1 + a_2 d_2)} > 0$

If $a_1 N_0 (a_1 d_1 + a_2 d_2) > [a_2 c_1 (a_1 d_1 + a_2 d_2) + a_1^2 d_1 T_0]$

One root (out of four roots) given by characteristic equation corresponding to equilibrium point E_2 is $\mu = b_1 \bar{\bar{R}} - b_2$. So, by Routh-Hurwitz criteria, E_2 is locally asymptotically stable if $\bar{\bar{R}} < \frac{b_2}{b_1}$ which implies that equilibrium level of nutrient resources is less than fraction of natural decay rate of plant biomass to the specific rate of conversion of resources into plant biomass. Further it is observed that E_3 exist only when E_2 is unstable that is $\bar{\bar{R}} > \frac{b_2}{b_1}$and in this case plant biomass will die out.

(III) The non-zero interior equilibrium $E_3(N^* \neq 0, R^* \neq 0, B^* \neq 0, T^* \neq 0)$:

Here $R^* = \frac{b_2+KB}{b_1}, B^* = \frac{a_1N^*-a_2}{b_1}, T^* = \frac{T_0}{d_1+d_2N^*}, N^* = \frac{N_0+Ka_2R^*+Kb_2B^*+K_1B^{*2}}{c_1+a_1R^*+d_1T^*}$

$N^* \frac{-h_2+\sqrt{h_2^2-4h_1h_3}}{2h_1} > 0$ if $\sqrt{h_2^2-4h_1h_3} > h_2, -4h_1h_3 > 0$

Where $h_1 = b_1c_1d_2 + a_1b_2d_2 - Ka_1b_2d_2, h_2 = b_1c_1d_1 + a_1b_2d_1 + b_1d_1T_0 - Ka_1b_2d_1 - b_1d_2N_0 - Ka_2b_2d_2 + Ka_2b_2d_1, h_3 = -b_1d_1N_0$

Study of Interior Equilibrium and Local Hopf Bifurcation

The exponential characteristic equation about equilibrium E_3 is given by:

$$(\lambda^4 + m_1\lambda^3 + m_2\lambda^2 + m_3\lambda + m_4) + (n_1\lambda^3 + n_2\lambda^2 + n_3\lambda + n_4)e^{-\lambda\tau} = 0 \quad (5)$$

Here $m_1 = -(M_1 + M_6 + M_{11} + M_{16})$

$$m_2 = (M_1M_{16} + M_1M_{11} + M_1M_{16} + M_6M_{11} + M_6M_{16} + M_{11}M_{16} - M_{12}M_{15} - R^*a_1M_9)$$

$$m_3 = -(M_1M_6M_{11} + M_1M_{11}M_{16} + M_6M_{11}M_{16} - M_1M_{12}M_{15} - M_6M_{12}M_{15} - R^*a_1M_6M_9 - R^*a_1M_9M_{16})$$

$$m_4 = (M_1M_6M_{11}M_{16} - M_1M_6M_{12}M_{15} - R^*a_1M_6M_9M_{16})$$

$$n_1 = M_5, n_2 = -(M_5M_6 + M_5M_{16})$$

$$n_3 = (M_5M_6M_{11} + M_5M_6M_{16} + R^*a_1M_5M_{10} - M_5M_{12}M_{15})$$

$$n_4 = (M_5M_6M_{11}M_{16} - M_5M_6M_{12}M_{15} - R^*a_1M_5M_{10}M_{16})$$

Where $M_1 = (a_1N^* - a_2), M_5 = b_1B^*, M_6 = -(b_2 + 2KB^*), M_9 = (Ka_2 - a_1N^*)$

$$M_{10} = (Kb_2 + 2KB^*), M_{11} = -(c_1 + a_1R^* + d_2T^*), M_{15} = -d_2T^*, M_{16} = -(d_1 + d_2N^*)$$

Clearly $\lambda = i\omega$ is a root of equation (5) if it satisfies equation (5).

$$(\omega^4 - im_1\omega^3 - m_2\omega^2 + im_3\omega + m_4) = (in_1\omega^3 + n_2\omega^2 - in_3\omega - n_4)(cos\, \omega\tau - isin\, \omega\tau)$$

Separating real and imaginary parts, we get:

$$\omega^4 - m_2\omega^2 + m_4 = (n_2\omega^2 - n_4)\cos\omega\tau + (n_1\omega^3 - n_3\omega)\sin\omega\tau \quad (6)$$

$$m_3\omega - m_1\omega^3 = (n_1\omega^3 - n_3\omega)\cos\omega\tau - (n_2\omega^2 - n_4)\sin\omega\tau \quad (7)$$

Squaring and adding equations (6) and (7), we get:

$$\omega^8 + p\omega^6 + q\omega^4 + r\omega^2 + s = 0 \quad (8)$$

Where $p = (m_1^2 - 2m_2 - n_1^2), q = (m_2^2 - 2m_1m_3 + 2n_1n_3 + 2m_4 - n_2^2),$

$$r = (m_3^2 - 2m_2m_4 + 2n_2n_4 - n_3^2), s = (m_4^2 - n_4^2)$$

Putting $\omega^2 = y$ in equation (8), we get:

$$y^4 + py^3 + qy^2 + ry + s = 0 \quad (9)$$

Lemma 2. If $s < 0$, Equation (9) contains at least one positive real root.

Proof. Let $h(y) = y^4 + py^3 + qy^2 + ry + s$

Here $h(0) = s < 0$, $\lim_{y\to\infty} h(y) = \infty$

So, $\exists y_0 \in (0, \infty)$ such that $h(y_0) = 0$

Proof completed.

Also $h'(y) = 4y^3 + 3py^2 + 2qy + r$

$$h'(y) = 0 \Longrightarrow 4y^3 + 3py^2 + 2qy + r = 0 \tag{10}$$

Let $x = y + \frac{3p}{4}$, $a = \frac{q}{2} - \frac{3p^2}{16}$, $b = \frac{r}{4} - \frac{pq}{8} + \frac{p^3}{32}$, $D = \left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3$ and $\sigma = \frac{-1+\sqrt{3}i}{2}$

Equation (10) becomes: $x^3 + ax + b = 0$ (11)

Three roots of equation (11) come out to be:

$x_1 = \left(-\frac{b}{2} + \sqrt{D}\right)^{1/3} + \left(-\frac{b}{2} - \sqrt{D}\right)^{1/3}$, $x_2 = \left(-\frac{b}{2} + \sqrt{D}\,\sigma\right)^{1/3} + \left(-\frac{b}{2} - \sqrt{D}\sigma^2\right)^{1/3}$,

$x_3 = \left(-\frac{b}{2} + \sqrt{D}\sigma^2\right)^{1/3} + \left(-\frac{b}{2} - \sqrt{D}\,\sigma\right)^{1/3}$ such that $y_i = x_i - \frac{3p}{4}$, $i = 1,2,3$.

Lemma 3. Suppose $s \geq 0$

(I) If $D \geq 0$, then equation (9) has positive roots iff $y_1 > 0$, $h(y_1) < 0$

(II) If $D < 0$, then equation (9) has positive roots iff there exists at least one$y^* \in (y_1, y_2, y_3)$ such that $y^* > 0$ and $h(y^*) \leq 0$.

(III) If $D < 0$,we know that equation (11) possesses only three roots x_1, x_2, x_3 which means that equation (10) has three roots y_1, y_2, y_3 and at least one of them is real.

Proof. (I)If $D \geq 0$, then equation (11) possesses a unique real rootx_1, which means equation (10) possesses unique real root y_1.

As $h(y)$ is a differentiable function and $\lim_{y\to\infty} h(y) = \infty$, we have $y_1$as the unique critical point of $h(y)$ which happens to be the minimum point of $h(y)$.

Suppose equation (9) possesses positive roots. In general, we suppose that it has 4 positive roots denoted by$y^*{}_i$, $i = 1,2,3,4.$, and then equation (8) has 4 positive roots $\omega_i = \sqrt{y^*{}_i}$, $i = 1,2,3,4$.

From equation (7) and (8), we get:

$$\cos\omega\tau = \frac{(\omega^4 - m_2\omega^2 + m_4)(n_2\omega^2 - n_4) + (n_3\omega - n_1\omega^3)(m_1\omega^3 - m_3\omega)}{(n_2\omega^2 - n_4)^2 + (n_3\omega - n_1\omega^3)^2}$$

So, if we conclude that

$$\tau_k{}^{(j)} = \frac{1}{\omega_k}\left[cos^{-1}\left(\frac{(\omega^4 - m_2\omega^2 + m_4)(n_2\omega^2 - n_4) + (n_3\omega - n_1\omega^3)(m_1\omega^3 - m_3\omega)}{(n_2\omega^2 - n_4)^2 + (n_3\omega - n_1\omega^3)^2}\right) + 2j\pi\right]$$

where $k = 1,2,3,4.$; $j = 1,2 - -$,then $\mp i\omega_k$ is a pair of purely imaginary roots of equation (5)

We define $\tau_0 = \tau_{k_0}{}^{(j_0)} = \min_{1\leq k\leq 4, j\geq 1}\left[\tau_k{}^{(j)}\right]$, $\omega_0 = \omega_{k_0}$, $y_0 = y_{k_0}{}^*$.

Next, we need to show: $\frac{d}{d\tau}(Re\lambda(\tau_k)) > 0$.

This guarantees at least one Eigen value with positive real part for $\tau > \tau_k$.

Let $\lambda(\tau) = \psi(\tau) + i\omega(\tau)$ (12)
be the roots of equation (5) satisfying: $\psi(\tau_0) = 0,\ \omega(\tau_0) = \omega_0$

Substituting $\lambda(\tau)$ into equation (5) and differentiating both sides with respect to τ, we get:

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \left[\frac{(4\lambda^3 + 3m_1\lambda^2 + 2m_2\lambda + m_3) + (3n_1\lambda^2 + 2n_2\lambda + n_3)e^{-\lambda\tau}}{\lambda(n_1\lambda^3 + n_2\lambda^2 + n_3\lambda + n_4)e^{-\lambda\tau}} - \frac{\tau}{\lambda}\right]$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \left[\frac{(4\lambda^3 + 3m_1\lambda^2 + 2m_2\lambda + m_3)}{\lambda(n_1\lambda^3 + n_2\lambda^2 + n_3\lambda + n_4)e^{-\lambda\tau}} + \frac{(3n_1\lambda^2 + 2n_2\lambda + n_3)}{\lambda(n_1\lambda^3 + n_2\lambda^2 + n_3\lambda + n_4)} - \frac{\tau}{\lambda}\right]$$

So, Sign $\left[\frac{d(Re\lambda)}{d\tau}\right]$ = Sign $\left[Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]$

$$= \text{Sign}\left[Re\left(\frac{(4\lambda^3+3m_1\lambda^2+2m_2\lambda+m_3)}{\lambda(n_1\lambda^3+n_2\lambda^2+n_3\lambda+n_4)e^{-\lambda\tau}}\right)_{\lambda=i\omega} + Re\left(\frac{(3n_1\lambda^2+2n_2\lambda+n_3)}{\lambda(n_1\lambda^3+n_2\lambda^2+n_3\lambda+n_4)}\right)_{\lambda=i\omega}\right]$$

$$= \frac{\omega_k^2[4\omega_k^6 + 3(m_1^2 - 2m_2 - n_1^2)\omega_k^4 + 2(m_2^2 - 2m_1m_3 + 2n_1n_3 + 2m_4 - n_2^2)\omega_k^2 + (m_3^2 - 2m_2m_4 + 2n_2n_4 - n_3^2)}{\rho}$$

$= \frac{\omega_k^2 h'(y_k)}{\rho}$ where $\rho = \omega_k^2[(n_2\omega^2 - n_4)^2 + (n_3\omega - n_1\omega^3)^2]$

Thus, we have Sign$\left[\frac{d(Re\lambda)}{d\tau}\right]_{\tau=\tau_k^{j_0}}$ = Sign $\left[Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_k^{j_0}}$ = Sign $\left[\frac{\omega_k^2 h'(y_k)}{\rho}\right]$

Moreover, since $y_k > 0$ and $\rho > 0$, we conclude that $\left(\frac{dRe\lambda(\tau)}{d\tau}\right)_{\tau=\tau_k^j}$ and $h'(y_k)$ have same sign.

Thus, there exists at least one root of equation (5) which has positive real part and hence, interior equilibrium E_3 is unstable for $\tau > \tau_k$.

Theorem 1. When $\tau = 0$,then characteristic equation (5) becomes:

$$\lambda^4 + (m_1 + n_1)\lambda^3 + (m_2 + n_2)\lambda^2 + (m_3 + n_3)\lambda + (m_4 + n_4) = 0$$

This can be written as: $\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta = 0$ (13)

Here $\alpha = (m_1 + n_1),\ \beta = (m_2 + n_2),\ \gamma = (m_3 + n_3), \delta = (m_4 + n_4)$

By Routh-Hurwitz criteria, the interior equilibrium E_3 is locally asymptotically stable if and only if the following inequality holds:

$\beta\gamma > \delta$ and $\alpha(\beta\gamma - \alpha) > \gamma^2\delta$

Existing Experimental data on Growth of Brassica Juncea Under Toxic Metals

Model is also verified using already existing experimental data for growth of Green vegetables (Brassica juncea L. Czern) under the effect of toxic metals (Cu N.X.2015).

Effect of Cu on the growth of Brassica juncea:

Added Cu: ppm (g/Kg)	Plant Height		Yield	
	cm	%	g/pot	%
0 (0)	19.5	100	70.3	100
50 (0.05)	15.0	77	55.9	80
100 (0.1)	14.7	75	50.0	71
200 (0.2)	12.5	64	33.1	47

Table1. Effect of added Cu on growth of Brassica juncea (fresh weight).

Effect of Pb on the growth of Brassica juncea:

Added Pb: ppm (g/Kg)	Plant Height		Yield	
	cm	%	g/pot	%
0 (0)	19.5	100	70.3	100
50 (0.05)	14.7	75	37.3	53
100 (0.1)	13.4	69	34.7	49
200 (0.2)	12.5	56	31.0	44

Table2. Effect of added Pb on growth of Brassica juncea (fresh weight).

Effect of Zn on the growth of Brassica juncea:

Added Zn: ppm (g/Kg)	Plant Height		Yield	
	cm	%	g/pot	%
0 (0)	19.5	100	70.3	100
100 (0.1)	19.7	101	75.7	108
300 (0.1)	16.7	86	64.1	91
500 (0.5)	15.5	80	43.8	62

Table3. Effect of added Zn on growth of Brassica juncea (fresh weight).

Numerical Example

Here, the analytical results are supplemented graphically by numerical simulation with the help of MATLAB. We use the following set of parameter values:

$$T_0 = 1.9\,, N_0 = 1, c_1 = 0.1, a_1 = 1, d_2 = 0.1, b_1 = 0.13, a_2 = 0.3, b_2 = 0.2,$$

$$d_1 = 0.1, K_1 = 0.002, K = 0.002$$

With the initial values: $R(0) = 1, B(0) = 1, N(0) = 1, T(0) = 1$.

The values of variables in interior equilibrium turn out to be:

$$E_3(N^* = 1.5808, R^* = 2.7359, B^* = 0.6556, T^* = 4.0809)$$

Here, the value of uptake rate of toxic metal $d_2 = 0.1$ is taken as the one which lies in the actual range of values of toxic metals (0.0 to 0.2) for Cu, Pb and Zn as mentioned in above experimental data for Brassica Juncea. The following comparative study shows graphically the adverse effect of soil pollution on concentration of nutrients and density of plant biomass.

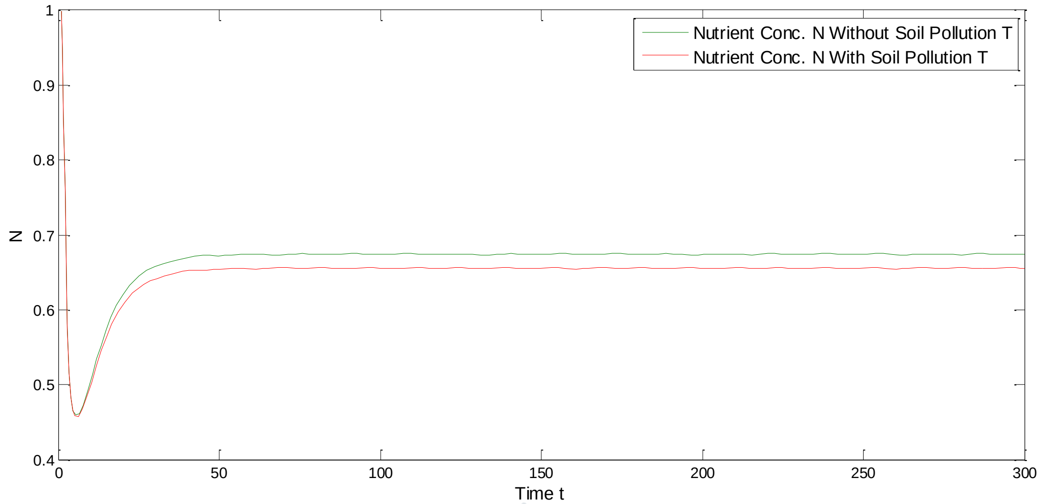

Figure 1. Trajectories showing adverse effect of toxic metal on nutrient concentration.

Figure 2. Trajectories showing adverse effect of toxic metal onplant biomass density.

The following simulation shows graphically how the system of equations (1) – (4) behaves differently for different values of delay parameter τ:

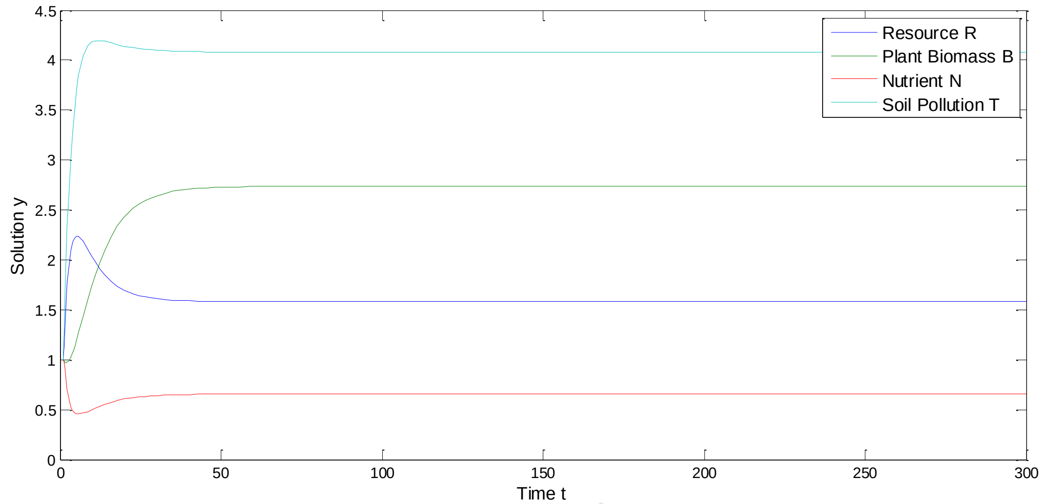

Figure 3. The interior equilibrium point $E_3(1.5808, 2.7359, 0.6556, 4.0809)$ of the system is stable when there is no delay that is $\tau = 0$.

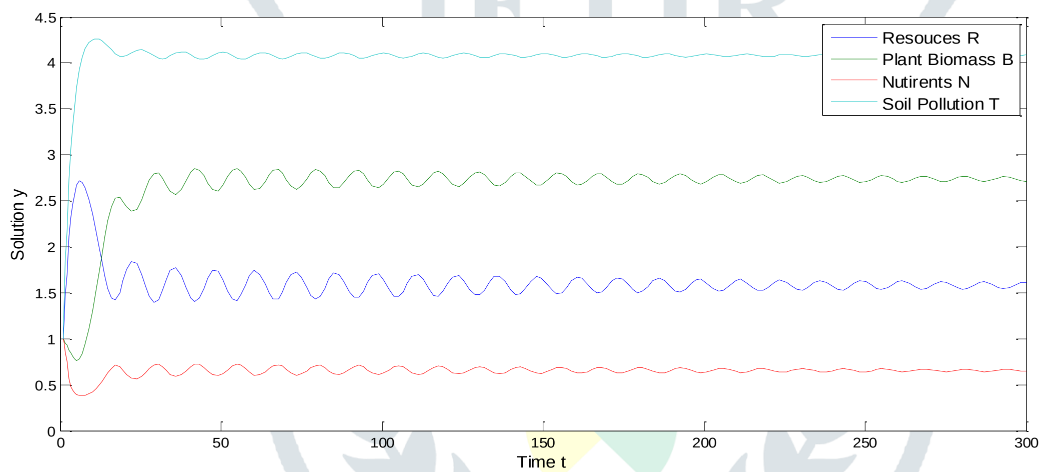

Figure 4. The interior equilibrium point $E_3(1.5808, 2.7359, 0.6556, 4.0809)$is asymptotically stablewhen delay $\tau < 3.53$

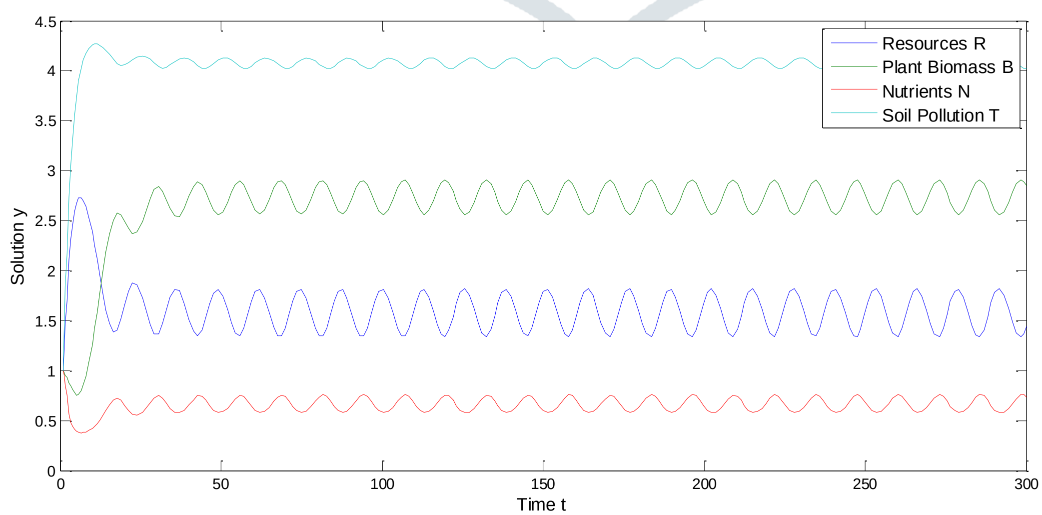

Figure 5. The interior equilibrium point $E_3(1.5808, 2.7359, 0.6556, 4.0809)$losses its stability and Hopf-bifurcation occurred when delay $\tau \geq 3.53$.

Sensitivity Analysis

All the parameters of the model governed by equations (1) – (4) are considered as constants. Estimation of the general sensitivity coefficients is done using the 'Direct Method'. For an instance, we consider parametera_1 and the partial derivatives of the solution (R, B, N, T) with respect to a_1 give rise to following set of sensitivity equations:

$$\frac{dS_1}{dt} = (a_1N - a_2)S_1 + a_1RS_3 - b_1BS_1(t-\tau) + NR \tag{14}$$

$$\frac{dS_2}{dt} = (-b_1 - 2KB)S_2 + b_1BS_1(t-\tau) \tag{15}$$

$$\frac{dS_3}{dt} = (Ka_2 - a_1N)S_1 + (Kb_2 + 2K_1B)S_2 - (c_1 + a_1N + d_2T)S_3 - d_2NS_4 \tag{16}$$

$$\frac{dS_4}{dt} = -d_2TS_3 - (d_1 + d_2N)S_4 \tag{17}$$

Where $S_1 = \frac{\partial R}{\partial a_1}, S_2 = \frac{\partial B}{\partial a_1}, S_3 = \frac{\partial N}{\partial a_1}, S_4 = \frac{\partial T}{\partial a_1}$

The details of the following sensitivity graphs have been discussed in conclusion section of the paper.

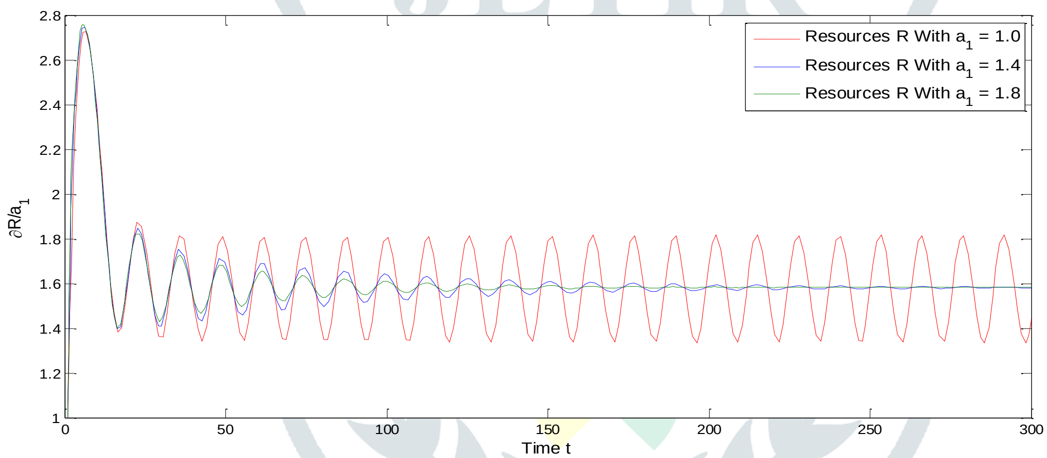

Figure 6. Time series graph between partial changes in R(density of favourable resources) and different values of parametera_1(interaction rate between nutrient and resources).

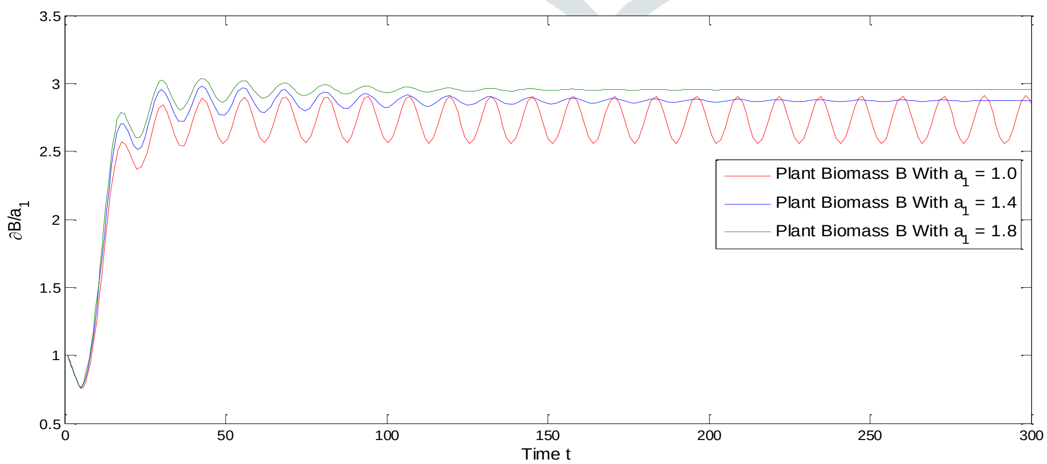

Figure 7. Time series graph between partial changes in B(density of biomass) and different values of parametera_1(interaction rate between nutrient and resources).

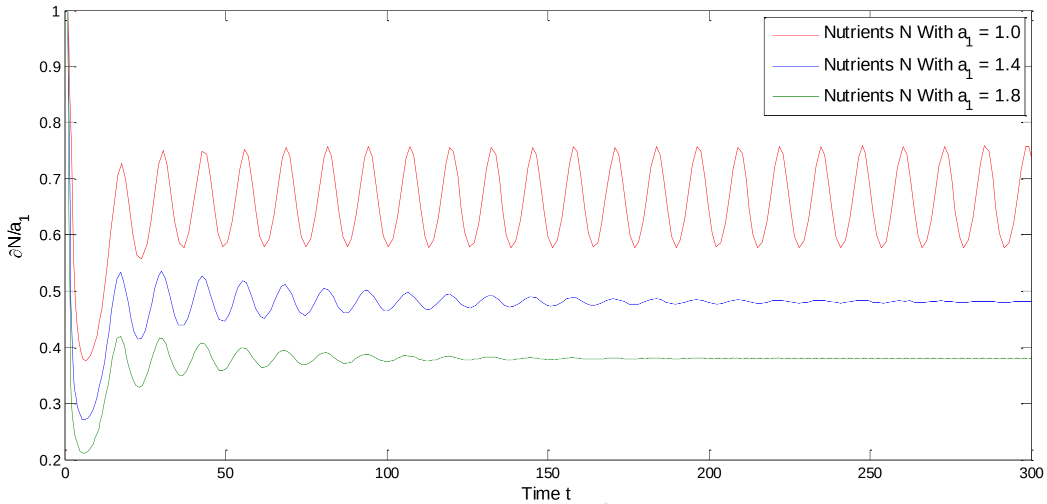

Figure 8. Time series graph between partial changes in N(concentration of nutrients) and different values of parametera_1(interaction rate between nutrient and resources).

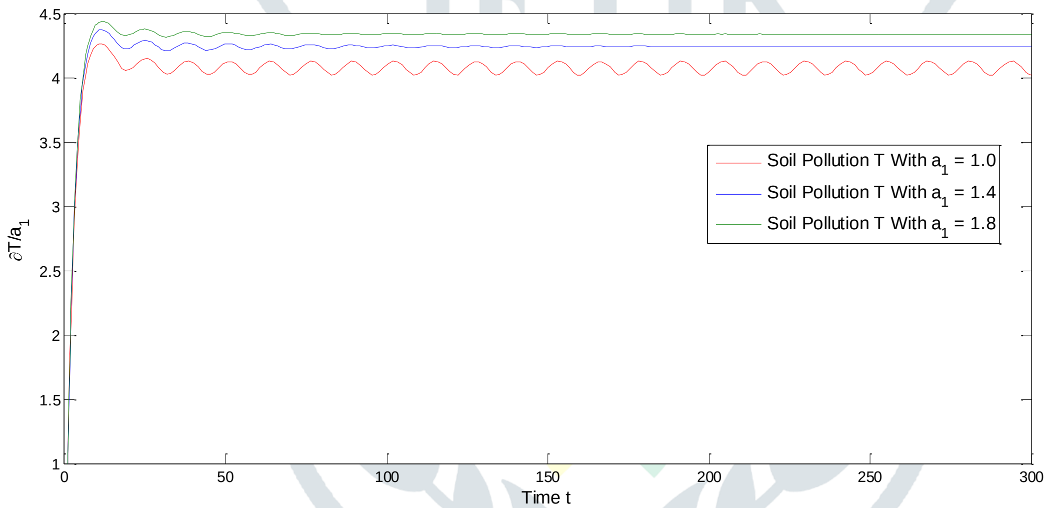

Figure 9. Time series graph between partial changes in T(concentration of soil pollution) and different values of parametera_1(interaction rate between nutrient and resources).

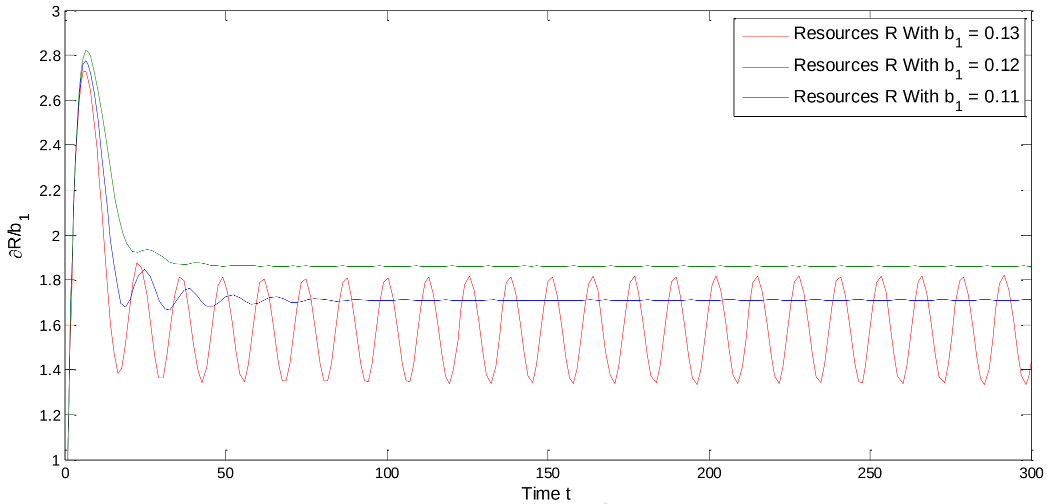

Figure 10. Time series graph between partial changes in R(density of favourable resources) and different values of parameterb_1(specific rate of utilization of resources by biomass).

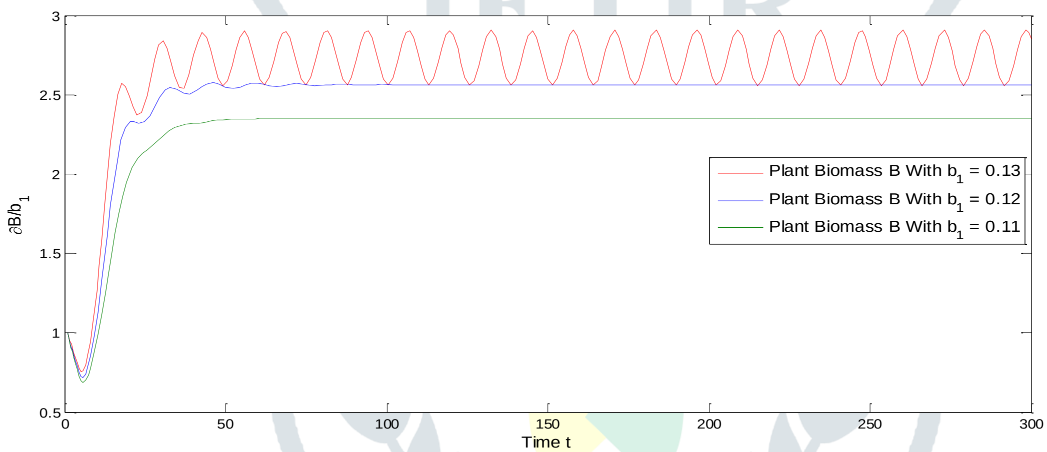

Figure 11.Time series graph between partial changes in B(density of plant biomass) and different values of parameterb_1(specific rate of utilization of resources by biomass).

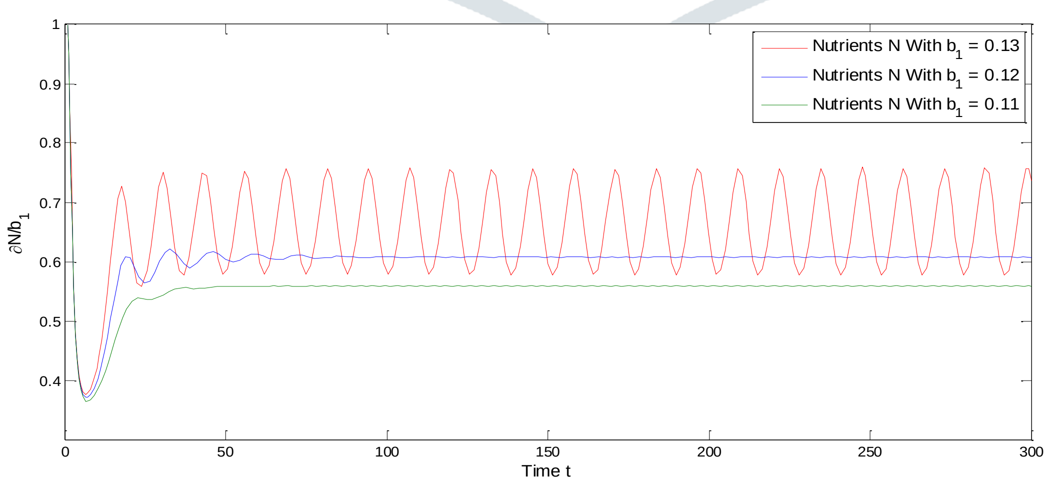

Figure 12.Time series graph between partial changes in N(concentration of nutrients) and different values of parameterb_1(specific rate of utilization of resources by biomass).

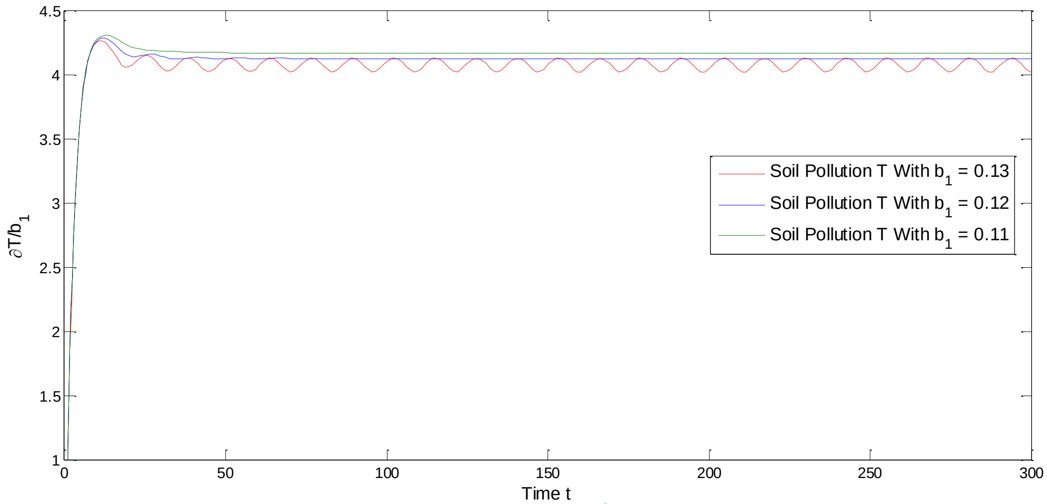

Figure 13. Time series graph between partial changes in T(concentration of soil pollution) and different values of parameterb_1(specific rate of utilization of resources by biomass).

Conclusion

In this paper, we have proposed a mathematical model to study the role of delay on the plant growth under the effect of soil pollution containing acid and toxic metal. It is also observed that the density of plant biomass undergoes a decrease when the rate of uptake of toxic metal is increased from $d_2 = 0$ to $d_2 = 0.1$ as shown by the Figure 2. This trend is similar to the one that is shown by real experimental data for Brassica Juncea where plant yield decreases when metal intake of Cu, Pb and Zn increase from 0.05 to 0.2 .We studied the stability and Hopf- bifurcation about the non-zero feasible interior equilibrium.It is concluded that in the absence of delay, interior equilibrium $E_3(N^* = 1.5808, R^* = 2.7359, B^* = 0.6556, T^* = 4.0809)$ is stable as shown by Figure 3. But for the same set of parameters there exist a critical value of the parameter delay below which the system is asymptotically stable as shown by Figure 4 and losses stability and becomes unstable above that value of parameter as shown by Figure 5. The system showsthe periodic oscillation when it passes through that critical value that is Hopf bifurcation occurs.

As we start Increasing the rate of interaction of nutrient and resources, the entire system starts convergingto stability. For $a_1 = 1$, the system i.e. the concentration of nutrients, the density of resources and plant biomass and concentration of soil pollution show Hopf bifurcation through periodic oscillations. But as we increase the value of a_1 from $a_1 = 1$ to $a_1 = 1.4$. , the system starts showing asymptotical stability as the periodic oscillations start dying down and eventually ends up converging to a stable equilibrium point as we further increase the value of $a_1$from $a_1 = 1.4$ to $a_1 = 1.8$. It has also been observed thatdensity of resources remain almost same throughout these increasing values ofa_1, but concentration of nutrient keep on decreasing with increase in the value of a_1. On the contrary, density of plant biomass and soil pollution show similar kind of increase as we increase the values of a_1.This phenomenon is shown by the Figures 6, 7, 8 and 9.

As we start decreasing the specific rate of utilization of delayed resources by plant biomass, the entire system starts converging to stability. For $b_1 = .13$, the system i.e. the concentration of nutrients, the amount of resources and plant biomass and soil pollution show Hopf bifurcation through periodic oscillations. But as we decrease the value of b_1 from $b_1 = .13$ to $b_1 = .11$., the system starts showing asymptotical stability as the periodic oscillations start dying down and eventually ends up converging to a stable equilibrium point as we further decrease the value of $b_1$from $b_1 = .11$ to $b_1 = .10$. It has also been observed that density of resources and soil pollution start increasing too with decrease in the value of b_1, but this increase is more visible in case of resources as compared to soil pollution. On the contrary, density of plant biomass and concentration of nutrients show similar kind of decrease in their values with decrease in the value of b_1. This phenomenon is graphically shown by Figures 10,11, 12 and 13.

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