Selective Pairwise Separation Properties as Productive Properties for Product of Two Bitopological Spaces

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Abstract. The motive of this paper is to investigate preservation of parwise separation properties for bispaces as namely $p - T_1$-bispace, $p - T_2$-bispace. $p -$regular bispace, $p -$completely regular bispace and $p - T_{3\frac{1}{2}}$ bispace under the cartesion products. Furthermore, we also investigate productive property of totally disconnected bispace.

Introduction and Preliminaries

As far as the development of bitopological spaces (briefly call bisapce) are concerned, in 1963 Kelly [2] introduced the bisapce and studies about non-symmetric functions for two arbitrary topologies on set. Further, in the same piece of work, the new concepts of pairwise Hausdorff, pairwise regular and pairwise normal, corresponding to the idea of separation axioms of topological space, are introduced in bisapce and thoroughly investigated. Patty [1], Weston [3], Reilly and Ivan [4], Pervin [9], Kim [8], Fletcher and Hoyle [7], Swart [5], Saegrove [6] and many other topologists carried out further research in the field of compactness, connectedness, total disconnectedness and more detailed separation properties in bitopological spaces.

In this paper, our main focus is on listing of selective separation properties and the concept of total disconnectedness for bisapce and to examine how these listed properties are productive properties under the product of two bisapces. First time, the idea of product of given family of bisapce is discussed independently by Swart [5] and Saegrove [6]. The solo motive of introducing product space is to obtain generalization of Tychnoff's theorem.

We use following notation in the paper. The triplet (X, τ_1, τ_2) a bisapce with two topologies τ_1 and τ_2 on X. The set $\tau_1 - C(A)$ and $\tau_2 - C(A)$ are closure of subset A of space X w.r.t. τ_1 and τ_2 respectively. The τ_1–open (τ_1–closed) and τ_2 –open (τ_2–closed) are open (closed) set in a bisapces X w.r.t. τ_1 and τ_2 respectively. The word pairwise is denoted by p so for example pairwise T_1 property is mentioned as as $p - T_1$. Now we present some important definitions and results from the literatures [2, 4, 6] that help to understand our main concepts.

A bisapce (X, τ_1, τ_2) is

1. $p - T_1$, when for arbitrary $x, y \in X$, $\exists$ sets G and H s.t. $x \in G$, $y \notin G$ and $y \in H, x \notin H$. Where x and y are distinct and G and H are open sets w.r.t. τ_1 and τ_2 respectively.
2. $p - T_2$, when for arbitrary $x, y \in X$, $\exists$ sets G and H s.t. $x \in G$, $y \in H$ and $G \cap H = \phi$. Where x and y are distinct and G and H are open sets w.r.t. τ_1 and τ_2 respectively.
3. regular w.r.t. τ_2 if $\forall x \in X$ and for $\forall \tau_1$ -closed set A s.t. $x \notin A$, $\exists$ sets G and H s.t. $x \in G, A \subseteq H$ with $G \cap H = \phi$. Where and G and H are open sets w.r.t. τ_1 and τ_2 respectively.
4. p −regular if τ_1 is regular w.r.t τ_2 and τ_2 is regular with respect to τ_1.
5. $p - T_3$ if X is p −regular and $p - T_1$.
6. p −completely regular if $\forall$ τ_1-closed set F_1 and $\forall a \notin F_1$, $\exists$ a p −continuous map $f : (X, \tau_1, \tau_2) \to (I, R, L)$ s.t. $f(a) = 1$ and $f(F_1) = \{0\}$, and $\forall$ τ_2 -closed set F_2 and for each point $b \notin F_2$, $\exists$ a p −continuous map $f_1 : (X, \tau_1, \tau_2) \to (I, R, L)$ s.t. $f_1(b) = 0$ and $f_1(F_2) = \{1\}$. Where $I = [0,1]$ is unit interval.
7. $p - T_{3\frac{1}{2}}$ if it is pairwise completely regular and $p - T_1$.

Definition [5]. The bispace $(\boldsymbol{X}, \boldsymbol{\tau_1}, \boldsymbol{\tau_2})$ is totally disconnected iff for arbitrary $\boldsymbol{x} \neq \boldsymbol{y} \in \boldsymbol{X}$, $\exists$ a $\boldsymbol{\tau_1}$ -open set $\boldsymbol{G}$ and a $\boldsymbol{\tau_2}$ -open set $\boldsymbol{H}$ s.t. $\boldsymbol{x} \in \boldsymbol{G}, \boldsymbol{y} \in \boldsymbol{H}$ with $\boldsymbol{X} = \boldsymbol{G} \cup \boldsymbol{H}$ and $\boldsymbol{G} \cap \boldsymbol{H} = \phi$.

Definition [6]. For a given family of bispace $\{(\boldsymbol{X_i}, \boldsymbol{\tau}_\alpha, \boldsymbol{\tau'}_\alpha)\}$, the product of this family of bispace is denoted by $\pi_\alpha(\boldsymbol{X}_\alpha, \boldsymbol{\tau}_\alpha, \boldsymbol{\tau'}_\alpha)$ which is also a bispace given by $(\boldsymbol{X}, \boldsymbol{\tau_1}, \boldsymbol{\tau_2})$. Where, $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$ are product topologies for bispace $\boldsymbol{X}$ for families $\{(\boldsymbol{X_\alpha}, \boldsymbol{\tau_\alpha})\}$ and $\{(\boldsymbol{X_\alpha}, \boldsymbol{\tau'_\alpha})\}$ respectively.

With the help of this definition, we are defining product of two bispace $(\boldsymbol{X}, \boldsymbol{\tau_1}, \boldsymbol{\tau_2})$ and $(\boldsymbol{Y}, \boldsymbol{\tau_3}, \boldsymbol{\tau_4})$ as $(\boldsymbol{X}, \boldsymbol{\tau_1}, \boldsymbol{\tau_2}) \times (\boldsymbol{Y}, \boldsymbol{\tau_3}, \boldsymbol{\tau_4}) = (\boldsymbol{X} \times \boldsymbol{Y}, \boldsymbol{\tau}, \boldsymbol{\tau'})$. Where, $\boldsymbol{\tau}$ is topology on $\boldsymbol{X} \times \boldsymbol{Y}$ with basis $\{\boldsymbol{G_1} \times \boldsymbol{G_3} : \boldsymbol{G_1} \in \boldsymbol{\tau_1} \text{ and } \boldsymbol{G_3} \in \boldsymbol{\tau_3}\}$ and $\boldsymbol{\tau'}$ is topology on $\boldsymbol{X} \times \boldsymbol{Y}$ with basis $\{\boldsymbol{G_2} \times \boldsymbol{G_4} : \boldsymbol{G_2} \in \boldsymbol{\tau_2} \text{ and } \boldsymbol{G_4} \in \boldsymbol{\tau_4}\}$.

Definition [6]. Any function f from (X, τ_1, τ_2) into (Y, τ'_1, τ'_2) is $p-$continuous if the induced functions from (X, τ_1) into (Y, τ'_1) and (X, τ_2) into (Y, τ'_2) are pair continuous.

Theorem[6]. For product topological space $\pi_\alpha(X_\alpha, \tau_\alpha, \tau'_\alpha)$, the projection map $P_\alpha : \pi_\alpha(X_\alpha, \tau_\alpha, \tau'_\alpha) \to (X_\alpha, \tau_\alpha, \tau'_\alpha)$ is pair onto, pair continuous and pair open.

Main Results

In this section we are going to investigate preservation of parwise separation properties for bispaces as namely $p - T_1$-bispace, $p - T_2$-bispace. $p-$regular bispace, $p-$completely regular bispace and $p - T_{3\frac{1}{2}}$ bispace under the cartesion products. Furthermore, we also investigate productive property of totally disconnected bispace.

Lemma 1. A bispace (X, τ_1, τ_2) is $p - T_1$ iff every singleton subset of (X, τ_1, τ_2) is a $p-$closed set.

Theorem 2. The p-T_1 bispace is closed w.r.t to cartesian product.

Proof. Consider two $p - T_1$ bispaces (X, τ_1, τ_2) and (Y, τ_3, τ_4). Let $(X, \tau_1, \tau_2) \times (Y, \tau_3, \tau_4) = (X \times Y, \tau, \tau')$. Choose any singleton subset $\{(x, y)\}$ from product bispace $(X \times Y, \tau, \tau')$. As bispace (X, τ_1, τ_2) and (Y, τ_3, τ_4) are pairwise T_1, therefore $\{x\}$ is a paiwise closed set in (X, τ_1, τ_2) and $\{y\}$ is $p-$closed in (Y, τ_3, τ_4). Consequently, $(X - \{x\}) \times (Y - \{y\})$ is a τ-open as well as τ'-open set. Evidently, $(X \times Y) - \{(x, y)\}$ is a τ-open as well as τ'-open set. Equivalently, $\{(x, y)\}$ is a τ-closed as well as τ'-closed set in $(X \times Y, \tau, \tau')$. Thus, product space is $p - T_1$.

Theorem 3. The p-T_2 bispace is closed w.r.t. to cartesian product.

Proof. Let (X, τ_1, τ_2) and (Y, τ_3, τ_4) are two p-T_2 bispacse with $(X, \tau_1, \tau_2) \times (Y, \tau_3, \tau_4) = (X \times Y, \tau, \tau')$. Consider two arbitrary distinct members (x_1, y_1) and (x_2, y_2) from $(X \times Y, \tau, \tau')$. Let us take $x_1 \neq x_2$. Now, $x_1 \neq x_2$ in a p-T_2 bispace space (X, τ_1, τ_2). Therefore, $\exists$ a τ_1-open set G_1 and a τ_2-open set G_2 s.t. $x_1 \in G_1$ and $x_2 \in G_2$with $G_1 \cap G_2 = \phi$. Evidently, $(x_1, y_1) \in G_1 \times Y$, a τ-open set in $X \times Y$. Also, $(x_2, y_2) \in G_2 \times Y$, a τ'-open set in $X \times Y$. Further, $(G_1 \times Y) \cap (G_2 \times Y) = (G_1 \cap G_2) \times Y = \phi$.

Theorem 4. The p −regular property is a productive property for product of two bispaces.

Proof. Suppose that two bispaces (X, τ_1, τ_2) and (Y, τ_3, τ_4) are p −regular. Let $(X \times Y, \tau, \tau')$ is product bispaces of given two bispaces, i.e. $(X, \tau_1, \tau_2) \times (Y, \tau_3, \tau_4) = (X \times Y, \tau, \tau')$. Consider arbitrary member (x, y) from product space $(X \times Y, \tau, \tau')$ and arbitrary τ-closed set F with $(x, y) \notin F$. *As* $(X \times Y) - F$ is τ-open and projection mappings $P_1: (X \times Y, \tau, \tau') \to (X, \tau_1, \tau_2)$ and $P_1: (X \times Y, \tau, \tau') \to (Y, \tau_3, \tau_4)$ are pair open, therefore $P_1((X \times Y) - F)$ is τ_1-open and $P_2((X \times Y) - F)$ is τ_3-open. Consequently, $X - P_1(F)$ is τ_1-open and $X - P_2(F)$ is τ_3-open. As $(x, y) \in (X \times Y) - F$, therefore $P_1(x, y) \in X - P_1(F)$ and $P_2(x, y) \in Y - P_2(F)$. Since, $P_1(F)$ is τ_1-closed and $x \notin P_1(F)$ and also (X, τ_1, τ_2) is p −regular. Therefore, $\exists$ a τ_1-open set G_1 and a τ_2-open set H_2 s.t. $x \in G_1$ and $P_1(F) \subseteq H_2$ with $G_1 \cap H_2 = \phi$. Similarly, $\exists$ a τ_3-open set G_3 and a τ_4-open set H_4 s.t. $y \in G_3$ and $P_2(F) \subseteq H_4$ with $G_3 \cap H_4 = \phi$. Then, $F \subseteq P_1 - (H_2)$ and $F \subseteq P_2 - (H_4)$ or $F \subseteq H_2 \times Y$ and $F \subseteq X \times H_4$. It means $F \subseteq (H_2 \times Y) \cap (X \times H_4) = H_2 \times H_4$, a τ'-open set. We see that $(x, y) \in G_1 \times G_3$, a τ-open set and $F \subseteq H_2 \times H_4$, a τ'-open set with $(G_1 \times G_3) \cap (H_2 \times H_4) = \phi$.

Remark 1. By using above theorem it can be prove that product of two $p - T_3$ bispace is also $p - T_3$.

Theorem 5. The p −completely regular property is a productive property for product of two bispaces.

Proof. Let (X, τ_1, τ_2) and (Y, τ_3, τ_4) are two p −completely regular bispaces. Suppose that $(X \times Y, \tau, \tau')$ is product bispaces of given two bispaces, i.e. $(X, \tau_1, \tau_2) \times (Y, \tau_3, \tau_4) = (X \times Y, \tau, \tau')$. Consider arbitrary member (x, y) from product space $(X \times Y, \tau, \tau')$ and arbitrary τ −closed set F with $(x, y) \notin F$. Since, $(x, y) \in (X \times Y) - F$, a τ-open set, therefore $\exists$ $G_1 \times G_3$, where G_1 is a τ_1-open set and G_3 is a τ_3-open set s.t. $(x, y) \in G_1 \times G_3 \subseteq (X \times Y) - F$. Evidently, $x \in G_1 \subseteq X$ or $x \notin X - G_1$, a τ_1-close set and (X, τ_1, τ_2)

is completely regular. Therefore, ∃ a p −continuous function $f_1: (X, \tau_1, \tau_2) \to ([0,1], R, L)$ s.t. $f_1(x) = 1$ and $f_1(X - G_1) = \{0\}$. Similarly, for $y \notin Y - G_3$, a τ_3-closed set, ∃ a p −continuous function $f_2: (Y, \tau_3, \tau_4) \to ([0,1], R, L)$ s.t. $f_2(y) = 1$ and $f_2(Y - G_3) = \{0\}$. Since, projection mappings $P_1: (X \times Y, \tau, \tau') \to (X, \tau_1, \tau_2)$ and $P_2: (X \times Y, \tau, \tau') \to (Y, \tau_3, \tau_4)$ are p −continuous. Therefore, $h_1 = f_1 o P_1: (X \times Y, \tau, \tau') \to ([0,1], R, L)$ and $h_2 = f_2 o P_2: (X \times Y, \tau, \tau') \to ([0,1], R, L)$ are p −continuous. Now, $h_1(x, y) = f_1(P_1(x, y)) = f_1(x) = 1$ and $h_2(x, y) = f_2(P_2(x, y)) = f_2(y) = 1$. Also, for any $(u, v) \in F$ means $(u, v) \notin G_1 \times G_3$, i.e., $u \notin G_1$ or $v \notin G_3$. Thus, $h_1(u, v) = f_1(P_1(u, v)) = f_1(u) = 0 \; or \; h_2(u, v) = f_1(P_2(u, v)) = f_2(v) = 0$. Choose $h = \min\{h_1, h_2\}$. Then, $h: (X \times Y, \tau, \tau') \to ([0,1], R, L)$ is p −continuous with $h(x, y) = 1$ and $h(u, v) = 0$ or $h(F) = \{0\}$. Similarly, we can prove the result by considering arbitrary member (x', y') from product space $(X \times Y, \tau, \tau')$ and arbitrary τ'-closed set F' with $(x, y) \notin F'$.

Remark 2. By using above theorem it can be proved that product of two $p - T_{3\frac{1}{2}}$ is also $-T_{3\frac{1}{2}}$.

Theorem 6. The totally disconnected bispace is closed w.r.t. cartesian product.

Proof. The spaces (X, τ_1, τ_2) and (Y, τ_3, τ_4) are two totally disconnected bispaces with $(X, \tau_1, \tau_2) \times (Y, \tau_3, \tau_4) = (X \times Y, \tau, \tau')$. Consider two arbitrary distinct members (x_1, y_1) and (x_2, y_2) from $(X \times Y, \tau, \tau')$. Choose $x_1 \neq x_2$. As, $x_1 \neq x_2$ in a totally disconnected bispaces space (X, τ_1, τ_2). So, ∃ a τ_1-open set G_1 and a τ_2-open set G_2 such that with $X = G_1 \cup G_2$ with $x_1 \in G_1, x_2 \in G_2$ and $G_1 \cap G_2 = \phi$. Evidently, $(x_1, y_1) \in G_1 \times Y$, a τ-open set in $X \times Y$. Also, $(x_2, y_2) \in G_2 \times Y$, a τ'-open set in $X \times Y$. Further, $(G_1 \times Y) \cup (G_2 \times Y) = (G_1 \cup G_2) \times Y = X \times Y$ and $(G_1 \times Y) \cap (G_2 \times Y) = (G_1 \cap G_2) \times Y = \phi$.

Conclusion. In this paper we study about pairwise separation properties namely, $p - T_1$, $p - T_2$, p −regular, p −completely regular, $p - T_{3\frac{1}{2}}$ for bispace and proved that these are preserve during the cartesion product of such corresponding spaces. Furthermore, totally disconnectedness is productive property under product of two bispaces.

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