A note on topological and semi-topological groups

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Abstract

The topological groups are dealing with the continuity of the group operations. In this paper, we gave a survey report on the topological and semi-topological groups. Some results are also given with explanation.

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1. Introduction

Definition 1.1: (Semi Topological Group) A triple $(G,\tau,*)$ is called as semi-topological group, where (G,τ) is a topological space and $(G,*)$ is a group, if the group operations $*: G\times G\to G$ that maps (x,y) to $x*y$ is continuous in each variable, i.e., the mapping $g_{y_o}: G\to G$ by $x\to x*\ y_0$ and $g_{x_o}: G\to G$ by $y\to x_0*y$ are is continuous for all y_0 in G and $x_0\in G$, respectively

Remark 1.1: A semi-topological group is a pair $(G,*)$ with a topology τ such that $\forall x,y\in G$ and each open set $W\ni x*y^{-1}$ there are open sets $U\ni x$ and $V\ni y$ such that $U*V^{-1}\subset W$.

Example 1.1: The $(R,+,\tau_\ell)$, where τ_ℓ is the lower limit topology, $\tau_\ell=\{[a,b): a,b\in R\}$ generated by a set $[a,b): a,b\in R$ and $a<b$. Here, the mapping $(x,y)\to xy$ is a continuous, but $x\to x^{-1}$ is not continuous.

Remark 1.2: For semi-topological and topological groups we have the following results:

(a) When we take group operation addition instead of multiplication then we replace $x.y$ by $x+y$ and x^{-1} by $-x$.

(b) Clearly by definition, every topological group is a semi topological, but converse is not true, in general.

Theorem 1.1: Let X and Y be two spaces. Suppose that $f: X\to Y$ is a continuous mapping. Then the following statements are equivalent:

(a) f is continuous.

(b) For each closed set C of Y, $f^{-1}(C)$ is closed in X.

(c) For each $x \in X$ and each neighborhood V of $f(x)$, $f^{-1}(V)$ is a neighborhood of x in X.

(d) For each $x \in X$ and each $V \ni f(x)$, $\exists\, U \ni x$ such that $f(U) \subset V$.

(e) For each $B \subset Y$, $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

(f) For each $A \subset X$, $\overline{f(A)} \subset f(\overline{A})$.

Definition 1.2: Let *G* be a semi topological group. Let *a* be any fixed element of *G*. Then the mappings $r_a: x \to xa$ and $\ell_a: x \to ax$ on *G* are called right and left translations respectively.

Theorem 1.2: Let *G* be a semi-topological group and $a \in G$. The functions from *G* to *G* are called right and left translations of *G* by the element *a* are given as

$$r_a : x \to xa \text{ and } \ell_a: x \to ax.$$

The translations of *G* give homeomorphism in each case.

Proof: In order to show that r_a is a homeomorphism. We first claim that r_a is bijective. Let $y \in G$, then we can write $r_a: y \to ya$. This mapping maps the element ya^{-1} to *y* by multiplying a^{-1} on the right both sides. So r_a is surjective, as every element of co domain is mapped to at least one element of the domain that is, image and co-domain are equal.

Now we claim that r_a is injective. Assume that $r_a(x) = r_a(y)$ for some $x, y \in G$.

This implies that $xa = ya$. Now by multiplying a^{-1} on the right, we get, $x = y$.

Hence, r_a is injective.

As *G* is a semi-topological group, then, it has continuous group operations in each variable separately.

Let *W* be a neighborhood of xa. Since *G* is a semi topological group then $\exists\, U \ni x$ such that $U_y \subset W$.

Because, function r_a fixes $a \in G$, so that we get $\forall\, W \in F_{xa}\ \exists\, U \in F_x$ such that $U_y \subset W$, this shows r_a is continuous.

Consider the inverse mapping $r_a^{-1}(x): x \to xa^{-1}$, which is equivalent to $r_a : xa \to x$. By same argument one can say r_a^{-1} is continuous. Hence, r_a is a homeomorphism. Similarly, we can show it for left translation ℓ_a.

2. Topological Group

In this section we discuss the general results concerning topological groups, translations in topological groups and neighborhood system of the identity of topological group.

Definition 2.2: Let (G, m) be a group and τ be a topology on G, then we say that (G, m, τ) or simply G is a if the two basic operations: $m : G \times G \to G$ defined by $(x, y) \to m(x,\ y) = x\, y$, and the inversion map: $i : G \to G$ defined by $x \to\ x^{-1}$ are continuous. Simply we can say if the function $G \times G \to G$, defined by $(x,\ y) \to x\, y^{-1}$ is continuous. Then G is a topological group.

Example 2.1: A group with discrete topology is a topological group.

Example 2.2: The set $R^* = R \setminus \{0\}$, multiplication as group operation and the topology induced from R.

Example 2.3: $R^+ = \{x \in R : x > 0\}$, with a topology induced from $R \setminus \{0\}$. This is the subgroup of $R \setminus \{0\}$.

Remark 2.1: The continuity of function depends not only upon the function *f* itself, but it also depends upon the topologies on the domain and range space. For example if the topology of range space *Y* is given by basis *B*. Then in order to prove the continuity of *f*, it is sufficient to show that inverse image of every basis element is open.

Theorem 2.1: Every topological group is a semi topological group, but converse is not true always.

Proof: By definition, we can say that every topological group is a semi topological group. The converse is not true in general.

Example 2.4: Let $G = R$, be an additive abelian group. Let *G* be a set having the topology which has a basis element of form $\{[a, b): -\infty < a \le x < b < \infty\}$. This means that the topology on *G* is a lower limit topology. Since 0 is a identity element of $G = R$ with respect to addition as binary operation. Clearly, for each neighbourhood $[a, b)$ of 0. We see that $[a, b/2)$ is also neighborhood of 0. It is suffices to show that $g(x, y) = x + y$ is continuous in each variable separately.

Let *U* and *V* are two subsets of a group *G* then,

$$U + V = \{x + y: x \in U\ and\ y \in V\} \text{ and } -U = \{-x: x \in U\}.$$

As *g* is continuous in *x* or *y* if and only if for each $W \ni x + y$ there exists a neighborhood $U \ni x$ or $V \ni y$ such that $Uy \subset W$ or $xV \subset W$.

Now, $[a, b)$ and $\left[a, \frac{b}{2}\right)$ are neighbourhoods of 0, so it follows that *g* is continuous in each variable separately at 0.

Hence, by definition G is a semi-topological group. We claim that the mapping $f: x \to x^{-1}$ is not continuous at 0. Since we know that f is continuous if for each neighborhood W of $-x$ such that there exists a neighborhood V of x such that $-V \subset W$.

If $[0, b)$ is a neighborhood of 0 then, there is no neighborhood V of 0 such that $-V \subset [0, b)$. Therefore, f is not continuous. Hence, G is not a topological group as desired.

Theorem 2.2: Let G be a topological group. Then the following statements are equivalent:

(i) G is a T_0-space

(ii) G is a T_1-space

(iii) G is a Hausdorff space.

Theorem 2.3: Let $f: X \to Y$ and $g: Y \to Z$ are continuous maps, then the map $gof: X \to Z$ is continuous.

Theorem 2.4: Let G be a topological group. Then the following maps $r_a : x \to xa$, $\ell_a: x \to a\, x$, $x \to x^{-1}$ and $x \to axa^{-1}$ are homeomorphisms.

Proof: (i) Given that G is a topological group. Our aim is to show that r_a is a homeomorphism. We first claim that r_a is bijective. Let $y \in G$, then $r_a: y \to ya$.

This mapping maps the element ya^{-1} to y by multiplying a^{-1} on the right both sides. So r_a is surjective, as every element of co domain is mapped to at least one element of the domain that is, image and co domain are equal. Now we claim that r_a is injective.

Assume that $r_a(x) = r_a(y)$ for some $x, y \in G$. This implies that $xa = ya$.

Now by multiplying a^{-1} on the right, we get, $x = y$. Hence, r_a is injective. Now, it remains to prove that r_a and its inverse are continuous.

For $W \ni xa$, there exists $U \ni x$ such that $Uy \subset W$. So that for each $W \in F_{xa}$ $\exists\, U \in F_x$ such that $U_y \subset W$, this shows r_a is continuous.

Consider the inverse mapping, $r_a^{-1}(x) : x \to xa^{-1}$, which is equivalent to $r_a: xa \to x$. By the similar argument, r_a^{-1} is continuous. Hence r_a is a homeomorphism by definition. Similarly we can show it for left translation ℓ_a .

(ii) The inverse mapping $x \to x^{-1}$ is bijective.

The mapping $x \to x^{-1}$ is continuous and clearly, the inverse map of the inverse map is itself, this means that $x^{-1} \to x$ is continuous. Therefore, by definition of homeomorphism the map $x \to x^{-1}$ is homeomorphism, as desired.

(iii) We want to show that the inner automorphism map $x \to axa^{-1}$ is homeomorphism.

The mapping $x \to axa^{-1}$ is a composition of two mappings $x \to ax$ and $x \to xa^{-1}$.

We already prove these two maps homeomorphism, therefore their composition $x \to axa^{-1}$ is also homeomorphism, because composition mapping is always continuous.

3. Neighborhood System

Definition 3.1: Let X be a topological space, let $x \in X$ be any arbitrary point. A subset V of X is said to a neighborhood of *x*, if there exists an open set U such that $x \in U \subset V$.

Example 3.1: The closed interval [0,1] is a neighborhood of 0.5, but not a neighborhood of 0.

Example 3.2: Consider set $X = \{p, q, r, s, t\}$ with $\tau_X = \big\{X, \phi, \{p, q, r\}, \{p, q, r, s\}, \{s\}\big\}$, then clearly $p, q \in X$ has exactly three neighbourhoods as given follows:

(a) $p, q \in A = \{p, q, r\} \in \tau_X$

(b) $p, q \in B = \{p, q, r, s\} \in \tau_X$

(c) $p, q \in C = X \in \tau_X$

So *p* and *q* has three open neighbourhoods.

Remark 3.1: Neighborhood of a point is not always an open set, clearly, every open set is a neighborhood of each of its point, because a set is called open if it is the neighborhood of every point.

Theorem 3.1: A subset A of X is open iff for each $x \in X, \exists\, V \ni x$ such that $V \subset A$.

Definition 3.2: Let (X, τ) be a space. Let $x \in X$ be any arbitrary point. Consider U be the set of all neighbourhoods of *x*. Let $V \subset U$, then V is said to be a *fundamental system of neighbourhoods* of *x*, if for each $U_1 \in U$ there exists some $V_1 \in V$ such that $V_1 \subset U$. we call V as a base for U.

Remark 3.2: Let X be a space, then for $x \in X$, let U denotes the set all neighbourhoods of *x*. Then following properties are established by using definitions of neighbourhoods and open set:

(a) For each U_1 in U , $x \in U_1$.

(b) If $U_1 \in U$ and W is any subset of X such that $U \subset W$, then $W \in U$.

(c) Every finite intersection of sets in U is also in U.

Definition 2.3: (Symmetric neighborhood) Let G be a topological group with identity e. A neighborhood V of e said to be symmetric if $V^{-1} = V$.

Definition 3.4: (Symmetric subset) Symmetric subset: Let G be group and S be any non-empty subset of G. then S is said to be symmetric if $S = S^{-1}$, where $S^{-1} = \{x^{-1}: x \in S\}$. In other words, S is symmetric if $x^{-1} \in S$, whenever $x \in S$.

Example 3.3: Let $G = R$, the intervals $(-a, a)$ with $a > 0$ are symmetric sets.

Example 3.4: The set $\{-1, 1\}$ is a symmetric set.

Remark 3.3: Let S be the symmetric subset of a topological group G. Then,

(a) The identity element e lies inside the subset, that is, $e \in S$.

(b) The inverse of any element of subset S of G also lies inside the subset, that is, if $x \in S$ then $x^{-1} \in S$.

Theorem 3.2: Let G be topological Group with identity e. Then there exists a fundamental system of neighbourhoods of identity element e of G in G.

Proof: We have given that G is topological group with identity e. Let A be the fundamental system of open neighbourhoods of identity e of G. Then our aim is to prove that there exists a symmetric subset A_1 of A that satisfies the condition of fundamental system of neighbourhoods. Let A_1 be the open set of A. Then for the group G be a topological group. Then the maps $r_a : x \to x\,a$, $l_a : x \to a\,x$ and $x \to a\,x\,a^{-1}$ are homeomorphisms: The third mapping is homeomorphism. Since $e \in G$, then the inverse mapping $e \to e^{-1}$ is also homeomorphism. As A is open neighborhood of e, then A^{-1} will be open neighborhood of e^{-1}. But we know the identity of a group is unique, so e^{-1}. Therefore, we can say that A and A^{-1} are two open neighbourhoods of identity element e.

Now assume that $S = A \cap A^{-1}$. Then,

$$S^{-1} = (A \cap A^{-1})^{-1} = A^{-1} \cap (A^{-1})^{-1} = A^{-1} \cap A = S.$$

This implies that $S^{-1} = S$.

So, S is symmetric neighborhood of x. Since A and A^{-1} are open, then so is their intersection S. We have $S = A \cap A^{-1}$ is the open neighborhood of e, because A ans A^{-1} are open neighbourhoods of e. Thus for each $A_1 \in A$, there is some S with $S \subset A$. The set of all such S forms a fundamental system of symmetric neighbourhoods of e.

Remark 2.4: Let G be a topological group with identity e. If A and B are subsets of G. Then we define the set $A * B$ as follows:

$A * B = \{a * b : a \in A, b \in B\}$ and $A^{-1} = \{x^{-1} : x \in A\}$.

Theorem 2.3: Let G be the topological group and e be its identity. Then, every neighborhood of identity e in the topological group contains the product of a symmetric neighborhood of identity with itself.

Proof: Let $(G, \tau, .)$ be a topological group, and e be its identity element. Suppose that U is the neighborhood of e. Claim that there exists a symmetric neighborhood V of the e such that $V.V \subset U$.

Let V be a symmetric neighborhood of e. By definition of symmetric neighborhood, $V = V^{-1}$, where $V^{-1} = \{v^{-1}: v \in V\}$. This implies that V^{-1} is neighborhood of e. Suppose that V is the open neighborhood of e contained in U. Note that $V.V^{-1}$ is an open set as V is an open set, then obviously $V.V^{-1}$ will be open neighborhood of e. Also $(V.V^{-1})^{-1} = V^{-1}.V$, implies that $V.V^{-1}$ is symmetric. Since, V and V^{-1} are open neighbourhoods of e contained in U. So that, $V.V^{-1} \subset U$.

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