On Divisor 3-Equitable Labeling of Certain Graphs

A. Parthiban[1],, G. Samdanielthompson[2]*

[1]Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab, India.

[2]Department of Mathematics, Hindustan College of Arts and Science, Padur, Kelambakkam, Chennai-603 103, India.

Abstract

An assignment of integers to the vertices of a graph G is called a vertex labeling of G. The concept of graph labeling plays a vital role in computer and network security. A graph G on n vertices is said to admit a divisor 3-equitable labeling if there exists a bijection $d : V(G) \to \{1, 2, \ldots, n\}$ defined by $d\,(e = xy) = \begin{cases} 1, if\, d(x)|d(y) \text{ or } d(y)|d(x) \\ 2, \quad if\ \frac{d(x)}{d(y)} = 2 \text{ or } \frac{d(y)}{d(x)} = 2 \\ 0, \qquad otherwise \end{cases}$ and $|e_d(i) - e_d(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph which permits a divisor 3-equitable labeling is called a divisor 3-equitable graph. A graph K_n in which "every pair of vertices is adjacent" is called a complete graph and a complete bipartite graph $K_{1,n}$ is known as a star graph. In this paper, we prove the non-existence of a divisor 3-equitable labeling of the complete graph K_n for $n \geq 5$ and star graph $K_{1,n}$ for $n \geq 6$.

Keywords: *3-equitable labeling, divisor 3-equitable labeling, complete graph, star graph*

1. Introduction

All graphs considered in this present investigation are finite, undirected, simple, and connected. By $G(V, E)$, or simply G, we mean a graph G with "vertex set V and edge set E." We also denote the "number of vertices in G" by $|V(G)|$ and the "number of edges in G" by $|E(G)|$. An assignment of integers to the vertices of G is called a vertex labeling of G. Graph labeling is a "strong communication" between the structure of graphs and number theory. Labeled graphs have a variety of applications in coding theory such as "missile guidance codes, design of good radar type codes, convolution codes with optimal auto correlation properties" etc. Labeled graphs also play a vital role in the field of X −ray crystallography, in determining optimal circuit layouts, and communication networks [4].

Definition 1. [2]
Let a and b be any two integers. If "a divides b", then that there is k (a "positive integer") such that $b = ka$ and denoted by $a \mid b$. If "a does not divide b", then it is denoted by $a \nmid b$.

Theorem 1. [2] (**The Division Algorithm**)
If a, b are integers with $b > 0$, then there exist "unique integers" q, r such that $a = q \cdot b + r$ with $0 \leq r < b$ where q is called the "quotient" and r is called the "remainder".

I. Cahit introduced the notion of "cordial labeling" in 1987. For a detailed study on cordial graphs and 3-equitable labeling, one can refer to [1, 2].

Definition 2. [3]
"A labeling (vertex) $f : V \to \{0, 1\}$ induces another labeling (edge) $f* : E \to \{0, 1\}$ defined by $f*(xy) = f(x) - f(y)|$. For $i \in \{0, 1\}$, let $v_f(i)$ and $e_f(i)$ be the number of vertices v and edges e with $f(v) = i$ and $f*(e) = i$, respectively. A graph G is cordial if there exists a labeling (vertex) f such that $v_f(0) - v_f(1) \leq 1$ and $e_f(0) - e_f(1) \leq 1$".

By merging the "divisibility concept in number theory and cordial labeling concept in graph labeling", Varatharajan et al. [6] introduced a new notion called "divisor cordial labeling" and proved various results. The definition of a divisor cordial labeling is given below.

Definition 3. [6]
Let $G = (V, E)$ be the given graph and $f : V(G) \to \{1, 2, \ldots, |V|\}$ be a bijective function. For each edge uv, assign the label "1" if either $f(u) \mid f(v)$ or $f(v) \mid f(u)$ and the label "0" otherwise. Then f is called a "divisor cordial labeling" if $|e_f(0) - e_f(1)| \leq 1$.

Further, in 2019, Sweta Srivastav and Sangeeta Gupta [5] introduced a new variant of graph labeling called a "divisor 3-equitable labeling" which is defined as follows.

Definition 4. [5]
A divisor 3-equitable labeling is a "bijection" $d : V(G) \to \{1, 2, \ldots, n\}$ defined by

$$d\,(e = xy) = \begin{cases} 1, if\, d(x)|d(y) \text{ or } d(y)|d(x) \\ 2, \quad if\ \frac{d(x)}{d(y)} = 2 \text{ or } \frac{d(y)}{d(x)} = 2 \\ 0, \qquad \text{otherwise} \end{cases}$$

such that $|e_d(i) - e_d\,(j)| \leq 1$ for all $0 \leq i, j \leq 2$, where $e_d(i)$ is the number of edges labeled with label "i". A graph which permits a divisor 3-equitable labeling is called a "divisor 3-equitable graph".

In this paper, we prove that the complete graph K_n for $n \geq 4$ and star graph $K_{1,n}$ for $n \geq 6$ do not admit a divisor 3-equitable labeling.

2. Main Results

In this section, first we recall a few important definitions and results concerning the divisor 3-equitable labeling of graphs which are relevant to the study undertaken. We also prove the non-existence of a divisor 3-equitable labeling of complete graph and star graph.

Theorem 2. [5]
The path P_n is a divisor 3-equitable graph.

Theorem 3. [5]
The cycle C_n is a divisor 3- equitable graph.

2.1. Divisor 3-Equitable Labeling of Complete Graphs

This section is devoted for proving the non-existence of the divisor 3-equitable labeling of complete graphs. First we recall the definition of a complete graph.

Definition 5.
The "complete graph K_n" is a graph in which "any two vertices are adjacent".

One can easily obtain the divisor 3-equitable labeling of $K_1, K_2, K_3,$ and K_4. One such example is given in Figure 1. So we consider K_n, for $n \geq 5$.

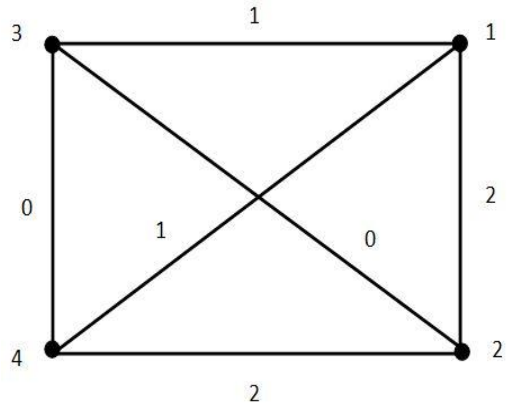

Figure 1. A divisor 3-equitable labeling of the complete graph K_4

Theorem 4.
A complete graph K_n does not permit a divisor 3-equitable labeling for all $n \geq 5$.

Proof.

Let $K_n = \{v_1, v_2, \dots, v_n\}$ be the given complete graph on $n \geq 5$ vertices. One can clearly observe that any two vertices in K_n are adjacent. In other words, a vertex $v \in K_n$ is adjacent to all other vertices of K_n. Now define a bijection $d : V(K_n) \to \{1, 2, \dots, n\}$ defined by $d\,(e = xy) = \begin{cases} 1, if\, d(x)|d(y) \text{ or } d(y)|d(x) \\ 2, \quad if\ \frac{d(x)}{d(y)} = 2 \text{ or } \frac{d(y)}{d(x)} = 2 \\ 0, \qquad otherwise \end{cases}$. We prove this theorem by a method of contradiction. Let's assume that $K_n, n \geq 5$ has a divisor 3-equitable labeling d such that "$|e_d(i) - e_d\,(j)| \leq 1$" for all $0 \leq i, j \leq 2$. Note that a vertex labeled with 1 in K_n, say $d(v_1) = 1$, is adjacent to all other vertices, $v_i: 2 \leq i \leq n$ labeled with $2, 3, \dots, n$ in any possible permutation. So as per the definition of a divisor 3-equitable labeling, if the label of a vertex v_1, say $d(v_1)$, divides the label of the adjacent vertex v_2, say $d(v_2)$, then the edge v_1v_2 must be given the label 1. As the vertex v_1 is adjacent to all other vertices in K_n, it gives 1 as the edge label to all other edges except the edge whose end vertex is labeled with 2, say v_2, (in which case the edge label is 2 as per the definition) in K_n which are incident on v_1. That is., $d(v_1v_i) = 1$ for all $3 \leq i \leq n$, a contradiction to the fact that $|e_d(i) - e_d(j)| \leq 1$ for all $0 \leq i, j \leq 2$. Therefore, a complete graph K_n does not permit a divisor 3-equitable labeling for all $n \geq 5$.

2.2. Divisor 3-Equitable Labeling of Star Graphs

This section is devoted to prove the non-existence of a divisor 3-equitable labeling of the star graphs.

Definition 6.

A star graph S_{n+1} is defined as a complete bipartite graph $K_{1,n}$.

One can clearly obtain the "divisor 3-equitable labeling of $K_{1,n}$, $1 \leq n \leq 5$." One such example is given in Figure 2.

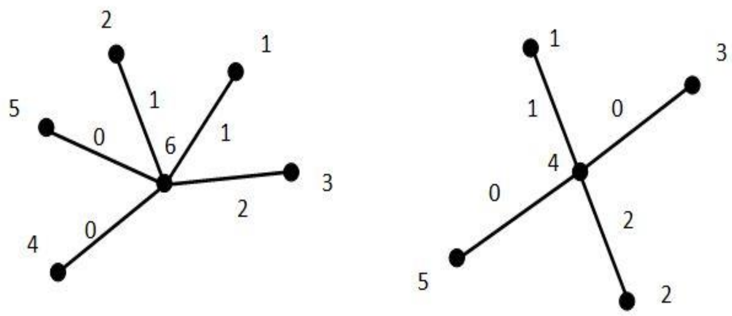

Figure 2. A divisor 3-equitable labeling of $K_{1,5}$ and $K_{1,4}$

Theorem 5.

A star graph $K_{1,n}$ does not permit a divisor 3-equitable labeling for all $n \geq 6$.

Proof.

Let $K_{1,n}$ be the given star graph on $n \geq 6$ vertices. We take $n = 6$ for the sake of discussion and so $|V(K_{1,6})| = 7$ and $|E(K_{1,6})| = 6$. One can note that there are six vertices of degree one and a vertex of degree 6 in $K_{1,6}$. We label the central vertex as v_0 and pendant vertices as $v_1, v_2, \dots, v_6$. Now define a bijective function $d : V\big(K_{1,6}\big) \to \{1, 2, \dots, 7\}$ defined by $d\,(e = xy) = \begin{cases} 1, if\, d(x)|d(y) \text{ or } d(y)|d(x) \\ 2, \quad if\ \frac{d(x)}{d(y)} = 2\ or\ \frac{d(y)}{d(x)} = 2 \\ 0, \qquad otherwise \end{cases}$. We prove by the method of contradiction. Let us assume that $K_{1,6}$ has a divisor 3-equitable labeling d with the property that $|e_d(i) - e_d\,(j)| \leq 1$ for all $0 \leq i, j \leq 2$. One can also observe that the number of edges labeled with label either 0 or 1 or 2 can be at most 2 (as there are exactly seven vertices in $K_{1,6}$) to satisfy the required divisor 3-equitable property $|e_d(i) - e_d(j)| \leq 1$ for all $0 \leq i, j \leq 2$. Now arises the following seven cases.

Case 1: When $d(v_0) = 1$

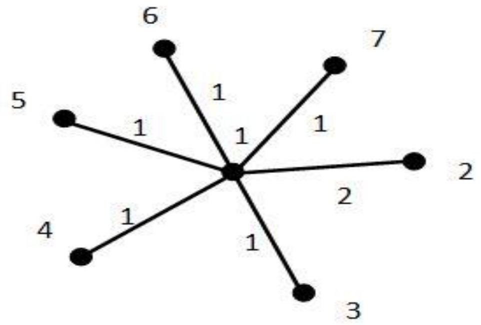

Figure 3. The central vertex in $K_{1,6}$ is labeled with 1

As the central vertex is adjacent to all other vertices and the label 1 divides all other labels, $d(v_0 v_i) = 1$ for all $2 \le i \le 6$ and $d(v_0 v_1) = 2$. So the number of edges labeled with 1 is 5, a contradiction.

Case 2: When $d(v_0) = 2$

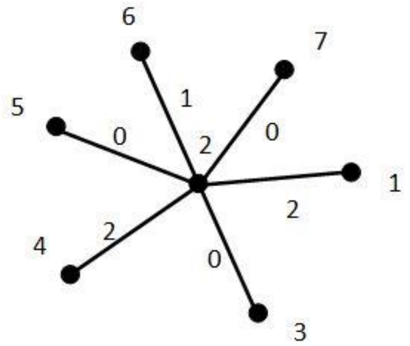

Figure 4. The central vertex in $K_{1,6}$ is labelled with 2

One can note that there are three edges with label 0, two edges with label 2, and only one vertex with label 1, a contradiction. This is because $|e_d(0) - e_d\,(2)| \le\ 1$ but $|e_d(0) - e_d\,(1)| = 2$.

Case 3: When $d(v_0) = 3$

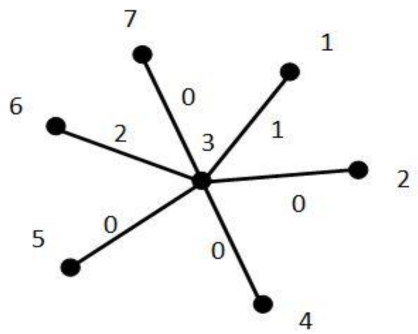

Figure 5. The central vertex in $K_{1,6}$ is labelled with 3

Interestingly there are four edges with label 0, one edge with label 1, and an edge with label 2, a contradiction as the number of edges labeled with 0 is more than 2.

Case 4: When $d(v_0) = 4$

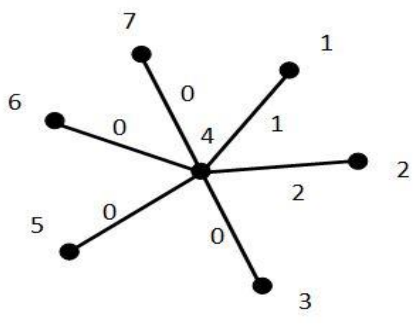

Figure 6. The central vertex in $K_{1,6}$ is labelled with 4

This is a similar case as Case 3.

Case 5: When $d(v_0) = 5$

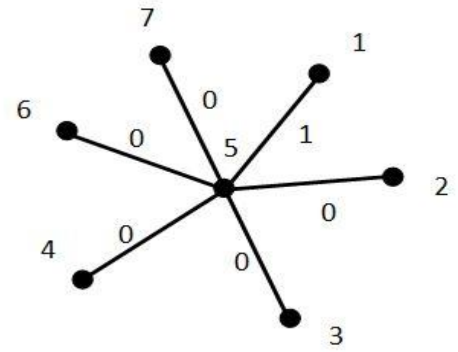

Figure 7. The central vertex in $K_{1,6}$ is labeled with 5

Interestingly there are five edges with label 0 and an edge with label 1, a contradiction.

Case 6: When $d(v_0) = 6$

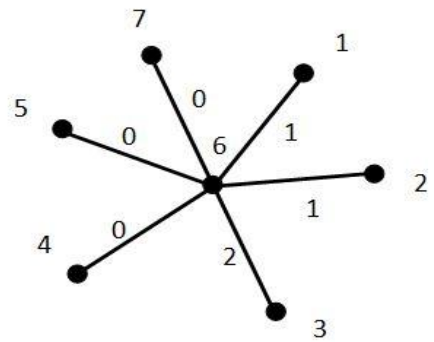

Figure 8. The central vertex in $K_{1,6}$ is labelled with 6

Now there are three edges with label 0, two edges with label 1, and an edge with label 2, again a contradiction because $|e_d(0) - e_d\ (1)| \leq\ 1$ but $|e_d(0) - e_d\ (2)| = 2$.

Case 7: When $d(v_0) = 7$

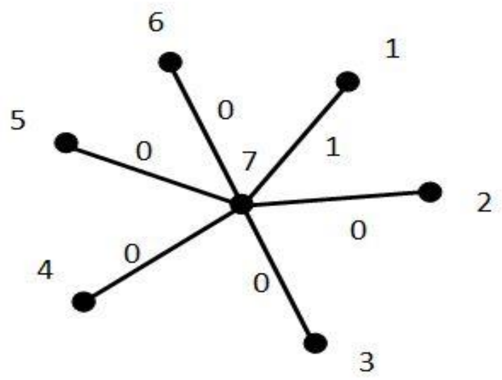

Figure 9. The central vertex in $K_{1,6}$ is labelled with 7

Here there are five edges of label 0 and an edge of label 1, a clear contradiction.
Hence $K_{1,n}$, when $n = 6$ does not admit a divisor 3-equitable labeling. A similar argument holds good for any $n \geq 7$.

3. Conclusion

The existence and non-existence of a divisor 3-equitable labeling of complete graphs and star graphs are established. Investigating divisor 3-equitable labeling of other classes of graphs is still open and this is for future research. We believe that the concept of divisor 3-equitable labeling may play a vital role in computer security. One can also explore the exclusive applications of divisor 3-equitable labeling in real life situations.

References

[1] I. Cahit, "Cordial Graphs: A Weaker Version of Graceful and Harmonious Graphs", Ars Combinatoria, vol. 23, (**1987**), pp. 201-207.
[2] I. Cahit, "On Cordial and 3-equitable Labelings of Graph", Utilitas Math., vol. 370, (**1990**), pp. 189-198.
[3] David Kuo, Gerard J. Chang, and Y. H. Harris Kwong, "Cordial Labeling of mK_n", Discrete Mathematics, vol. 169, (**1997**), pp. 121-131.
[4] Kenneth H. Rosen, "Elementary Number Theory and its Applications", sixth ed., Addison-Wesley, Reading, MA, (**2010**).
[5] R. Varatharajan, S. Navaneethakrishnan, and K. Nagarajan, "Divisor Cordial Labeling of Graphs", International Journal of Mathematical Combinatorics, vol. 4, (**2011**), pp. 15-25.
[6] D. B. West, "Introduction to Graph Theory", 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, (**2000**).