Equitable Power Domination Number of Generalized Petersen Graph, Balanced Binary Tree and Subdivision of Certain Graphs

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Abstract

Let G (V, E) be graph. A set $S \subseteq V$ is said to be a power dominating set (PDS) if every vertex $u \in V - S$ is observed by certain vertices in S by the following rules: (i) if a vertex v in G is in PDS, then it dominates itself and all the adjacent vertices of v and (ii) if an observed vertex v in G has $k > 1$ adjacent vertices and if $k-1$ of these vertices are already observed, then the remaining one non-observed vertex is also observed by v in $G.$ A power dominating set $S \subseteq V$ in $G\ (V, E)$ is said to be an equitable power dominating set (EPDS), if for every vertex $v \in V - S$ there exists an adjacent vertex $u \in S$ such that the difference between the degree of u and degree of v is less than or equal to 1, i.e., $|d(u) - d(v)| \leq 1$. The minimum cardinality of an equitable power dominating set of G is called the equitable power domination number of G, denoted by $\gamma_{epd}(G)$. "An edge is said to be subdivided if the edge xy is replaced by the path: xwy, where w is the new vertex. A graph obtained by subdividing each edge of a graph G is called subdivision of G, and is denoted by $S(G)$". In this paper we establish the equitable power domination number of subdivision of certain classes of graphs. We also obtain the equitable power domination number of the generalized Petersen graphs and balanced binary tree.

Keywords

Power dominating set, Power domination number, Equitable power dominating set, Equitable power domination number, Generalized Petersen graphs, Balanced binary tree, and Subdivision graph.

1. Introduction

Only simple, finite, undirected, and connected graphs are considered in this paper. A dominating set of a graph $G = (V, E)$ is a set S of vertices such that every vertex v in $V - S$ has at least one neighbor in S. The minimum cardinality of a dominating set of G is called the domination number of G, denoted by $\gamma_d(G)$ [8]. For a few other variants of dominating set refer to [9, 10].

A power dominating set $S \subseteq V$ in $G\ (V, E)$ is said to be an equitable power dominating set, if for every vertex $v \in V - S$ there exists an adjacent vertex $u \in S$ such that the difference between the degree of u and degree of v is less than or equal to 1, that is $|d(u) - d(v)| \leq 1$. The "minimum cardinality" of an equitable power dominating set of G is called the equitable power domination number of G, denoted by $\gamma_{\text{epd}}(G)$ [2]. For more results one can refer to [3, 4]. In this paper, we obtain the equitable power domination number of the generalized Petersen graphs and balanced binary tree.

2. Main Results

For the sake of convenience, by EPDS and EPDN we mean an equitable power dominating set and the equitable power domination number, respectively.

2.1 EPDN of the Generalized Petersen Graphs and Balanced Binary Tree

First we recall the definition of the generalized Petersen graph for the sake of completeness.

Definition 1 [1]

"The generalized Petersen graph $GP(n,k)$ is defined to be a graph with $V\left(GP(n,k)\right) = \{a_i, b_i: 0 \le i \le n-1\}$ and $E(GP(n,k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k}: 0 \le i \le n-1\}$, where the subscripts are expressed as integers modulo n $(n \ge 5)$ and k $(k \ge 1)$."

Note:

1. $GP(n,k)$ is isomorphic to $GP(n, n-k)$.
2. Without restriction of generality, one may consider the generalized Petersen graph $GP(n,k)$ with $k \le \lceil (n-1)/2 \rceil$.

Theorem 2

Let $GP(n,k)$ be the generalized Petersen graph.

Then $\gamma_{epd}(GP(n,k)) = \begin{cases} 2, & \text{for } k = 1, 2 \text{ and } m \ge 4 \\ 3, & \text{for m} \ge 10 \text{ and k} \ge 3. \end{cases}$

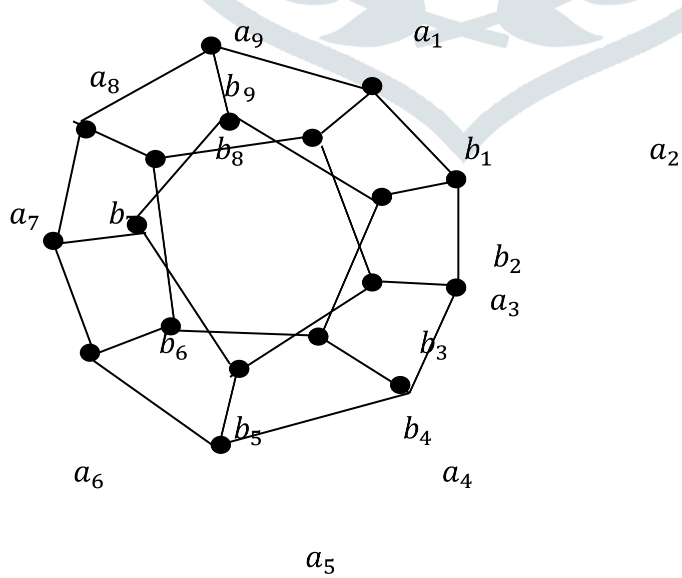

Fig. 1: $GP(9,2)$

Proof.

Let $GP(n,k)$ be the given GPG with $V = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}$ and edge set $E(GP(n,k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} : 0 \le i \le n-1\}$. To obtain the equitable power domination number of $GP(n,k)$, we consider the following two cases:

Case 1: For $k = 1, 2$ and $m \ge 4$

Without loss of generality, we choose any one of b_i's, $1 \le i \le n$ to be in S, say b_1. Note that b_1 equitably power dominates b_3, a_1, and b_{n-1}. Now the observed vertices b_3, a_1 and b_{n-1} have more than one non-observed vertices and so fail to observe their neighboring vertices which leads to choose another vertex to be in EPDS. Then one can choose either b_2 or b_n to be in S for the sake of minimum cardinality. Now it is easy to see that all the remaining non-observed vertices are observed by their respective neighbors and therefore $|S| = 2$.

Case 2: For m ≥ 10 and k ≥ 3
Construction of EPDS is similar to Case 1.

2.2 Equitable Power Domination Number of the Balanced Binary Tree

We recall a few relevant definitions needed for this section for the sake of convenience.

Definition 3 [5]

"A graph without cycles is called an acyclic graph and a connected acyclic graph is called as a tree."

Definition 4 [5]

A binary tree is a tree in which each vertex has at most 2 pendant vertices.

Definition 5 [5]

A balanced binary tree is a binary tree in which the left and right sub trees of every vertex differ in height by no more than one.

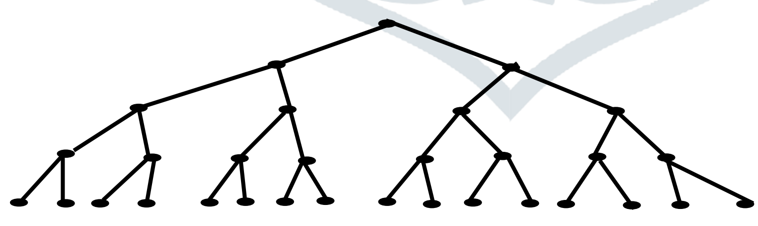

Fig.2: Balanced Binary Tree

Theorem 6

Let $B(1,k)$ be a balanced binary tree. Then $\gamma_{\text{epd}}(B(1,k)) = \sum_{n=0}^{n=k} 2^n - 2^{n-1}$.

Proof.

Let $B(1,k)$ be the given balanced binary tree on k levels with vertex set $V = \{a_0, a_1, a_2, a'_1, a'_2, a'_3, a'_4, a''_1, a''_2, a''_3, a''_4, a''_5, a''_6, a''_7, a''_8, \dots, a^n_1, a^n_2, a^n_3, a^n_4, a^n_5, a^n_6, a^n_7, a^n_8, \dots, a^n_n\}$

where $a_1^n, a_2^n, a_3^n, a_4^n, a_5^n, a_6^n, a_7^n, a_8^n, \dots, a_n^n$ are the pendant vertices. To obtain an equitable power dominating set S, without loss of generality, we choose a_0 to be in S. The vertex a_0 equitably power dominates a_1 and a_2. Now the vertices a_1 and a_2 have two non-observed vertices a_1', a_2' and a_3', a_4', respectively. So one has to choose any one between a_1 and a_2, say a_1, then a_2 is observed by a_0. Again as a_2 has two non-observed vertices a_3' and a_4', so one has to choose any one between a_3' and a_4', say a_3'. Also a_1 in S observes a_1' and a_2'. Proceeding in the same way, finally we need to choose $a_1^n, a_2^n, a_3^n, a_4^n, a_5^n, a_6^n, a_7^n, a_8^n, \dots, a_n^n$ as they are the pendent vertices and there are no adjacent vertices satisfying the desired equitable property. Thus we obtain the sequence of vertices, namely a_0, a_1, a_1', a_3', a_1'', a_3'', a_5'', a_7'', ... and so on. That is.,

$\gamma_{epd}(B(1,1)) = 1$

$\gamma_{epd}(B(1,2)) = 1 + 2$

$\gamma_{epd}(B(1,3)) = 1 + 2 + 2^3$

$\gamma_{epd}(B(1,4)) = 1 + 2 + 2^2 + 2^4$

$\gamma_{epd}(B(1,5)) = 1 + 2 + 2^2 + 2^3 + 2^5$

…

Thus $\gamma_{epd}(B(1,k)) = \sum_{n=0}^{n=k} 2^n - 2^{n-1}$.

2.2 Equitable Power Domination Number of Subdivision of Certain Classes of Graphs

The concept of subdivision in graphs was introduced by Trudeau, Richard J in 1993 [11]. We recall the definition of subdivision of a graph.

Definition 7 [11]

"An edge is said to be subdivided if the edge uv is replaced by the path:uwv, where w is the new vertex. A graph obtained by subdividing each edge of a graph G is called subdivision of the graph G, and is denoted by $S(G)$."

G **S(G)**

Fig.3: A graph G and subdivision of G, S(G)

Theorem 8

Let *G* be graph on *n* vertices. Then $\gamma_{epd}(S(G)) \geq \gamma_{epd}(G)$.

Proof.

Let G be the given graph with V = $\{v_1, v_2, \dots, v_n\}$ and edge set E = $\{e_1, e_2, \dots, e_n\}$. Obtain the subdivision of G, denoted $S(G)$, as follows: $V(S(G)) = V(G) \cup E(G)$ and $E(S(G)) = \{(v_i e_i), (e_i v_j)$: for $1 \leq i \leq n$ and $i + 1 \leq j \leq m - 1\}$. We consider the following two cases in obtaining an EPDS of S(G).

Case 1: For a vertex v_i incident with e_i for which $|d(v_i) - d(e_i)| \geq 1$ for at least one 'i'. Then one has to choose e_i to be in S. Thus $\gamma_{epd}(S(G)) \geq \gamma_{epd}(G)$.

Case 2: For a vertex v_i incident with e_i for which $|d(v_i) - d(e_i)| < 1$ for $1 \leq i \leq n$. Then S remains the same. Thus $\gamma_{epd}(S(G)) = \gamma_{epd}(G)$.

Theorem 9 [2]

Let C_n , $n \geq 3$ be a cycle. Then $\gamma_{epd}(C_n) = 1$.

Theorem 10

Let C_n , $n \geq 3$ be a cycle. Then $\gamma_{epd}(S(C_n)) = 1$.

Proof.

Let C_n be a cycle with $V(C_n) = \{v_1, v_2, \dots, v_n\}$. When one performs the subdivision on C_n, the resultant graph is again a cycle on $2n$ vertices. So by Theorem 9, $\gamma_{epd}(S(C_n)) = 1$.

Theorem 11 [2]

Let P_n , $n \geq 1$ be a path. Then $\gamma_{epd}(P_n) = 1$.

Theorem 12

Let P_n , $n \geq 3$ be a path. Then $\gamma_{epd}(S(P_n)) = 1$.

Proof.

Let P_n be a path with $V(P_n) = \{v_1, v_2, \dots, v_n\}$. An easy check shows that when one performs the subdivision on P_n, the resultant graph is again a path on $2n$ -1 vertices. So by Theorem 11, we deduce that $\gamma_{epd}(S(P_n)) = 1$.

Definition 13 [5]

"Any two distinct vertices of a graph G are adjacent then G is said to be complete graph and it is denoted by K_n."

Theorem 14 [2]

For a complete graph K_n, $\gamma_{epd}(K_n) = 1$.

Theorem 15

Let $S(K_n)$ be the subdivision of a complete graph K_n. Then $\gamma_{epd}(S(K_n)) = m + n$, for $n \geq 5$.

Proof.

Let K_n be a complete graph with $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and $E(K_n) = \{e_1, e_2, \dots, e_m\}$. By the definition of a complete graph, the degree of each vertex v_i, $d(v_i) = n - 1$ for $1 \leq i \leq n$. Obtain the subdivision of a complete graph K_n, denoted by $S(K_n)$ as follows: $V(S(K_n)) = V_1 \cup V_2$, where $V_1 = V(K_n) = \{v_1, v_2, \dots, v_n\}$ and $V_2 = E(K_n)$. One can notice that the subdivided graph of a complete graph K_n gives rise to the graph such that no two adjacent vertices with $|d(u) - d(v)| \leq 1$ and violate the equitable property. So to obtain an equitable power dominating set, one has to choose the entire vertex set to be in EPDS. Thus $|S| = m + n$.

Theorem 16 [2]

For a complete bipartite graph $K_{m,n}$, $m, n \geq 2$, γ_{epd} $(K_{m,n})$ = $\begin{cases} m+n, & if\ |m-n| \geq 2; \\ 2, & if\ |m-n| < 2. \end{cases}$

Theorem 17

For a complete bipartite graph $K_{m,n}$, $m, n \geq 4$,

$$\gamma_{epd}\ (S(K_{m,n})) = \begin{cases} mn+m+n, & if\ |m-n| \geq 2; \\ 1, & \text{otherwise.} \end{cases}$$

Proof.

Let $K_{m,n}$ be the given complete bipartite graph with $V\ (K_{m,n}) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \dots, u_m\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$ be the two partition sets of $K_{m,n}$. When we construct the subdivision of $K_{m,n}$, the number of vertices of $S(K_{m,n})$ becomes $m + n + mn$. Then the following two cases arise.

Case (i) $|m-n| \geq 2$

It is clear that $d(u_i) = n$ for every u_i in V_1 and $d(v_i) = m$ for every v_i in V_2. And the degree of newly added vertices is two. Since $|m-n| \geq 2$, the equitable property does not hold well between any two adjacent vertices. Then the entire vertex set of $S(K_{m,n})$ must be chosen to form the equitable power domination set.

Case (ii) $|m-n| < 2$

Let $S = \{u_1\}$. Choosing one vertex from one of the partition with degree more than one is enough to get an equitable power dominating set S. Therefore γ_{epd} $(K_{m,n})$ =1, whenever $|m-n| < 2$.

Note:

γ_{epd} $(S(K_{2,2}))$ = γ_{epd} $(S(K_{3,3}))$ = 1.

Definition 18 [5]

"The wheel graph with n spokes, $W_{1,n}$ is the graph that consists of a cycle C_n and one additional vertex, say u, that is adjacent to all the vertices of the cycle C_n."

Theorem 19 [2]

For a wheel graph $W_{1,n}$, $n \geq 5$, γ_{epd} $(W_{1,n}) = 2$.

Theorem 20

Let $W_{1,n}$, $n \geq 3$ be a wheel graph. Then γ_{epd} $(S(W_{1,n})) = n + 1$.

Proof.

Let $W_{1,n}$ be the given wheel graph on '$n + 1$' vertices with V $(W_{1,n})$ = $\{v_0, v_1, v_2, \dots, v_n\}$ where v_0 is the central vertex (the hub) and $v_i : 1 \leq i \leq n$ are the rim vertices & E $(W_{1,n})$ = $\{v_1', v_2', \dots, v_n'\ u_1, u_2, \dots, u_n\}$, where $u_1, u_2, \dots, u_n$ represent the spokes of $W_{1,n}$. It is interesting to note that the number of vertices in the subdivision of wheel graph, $S(W_{1,n})$ is $3n + 1$. Now to obtain an equitable power dominating set S, one has to choose the central vertex which is of maximum degree and no adjacent vertices equitably power dominate with it. Also from the remaining vertices, without loss of generality, choose v_1 to be in S as v_1 equitably power dominates v_1', v_n' and u_1. For v_1' and v_n' the only non-observed vertices are v_2 and v_n, respectively and hence are observed. For v_2 and v_n there are two non-observed vertices (one in the rim and another in the spoke). So we have to choose any one of the vertices, for the sake of minimum cardinality, we choose u_2 and u_n to be in S. And proceeding thus, one has to choose all the vertices in the spokes of $W_{1,n}$. Thus $S = \{ v_0, u_1, u_2, \dots, u_n\}$ and |S| = n + 1.

Definition 21 [5]

"The gear graph G_n is obtained from a wheel graph $W_{1,n}$ by subdividing each edge of the outer n −cycle of $W_{1,n}$ just once."

Note:

1. Gear graph G_n has $2n + 1$ vertices.
2. $\gamma_{epd}(G_3) = \gamma_{epd}(G_4) = 1$.

Definition 22 [5]

"Let P_n be a path. Then the n −ladder graph is defined as $P_2 \times P_n$."

We label the vertices of the first and second copy of P_n as $\{v_1, v_2, \dots, v_n\}$.and $\{v_1', v_2', \dots, v_n'\}$respectively. We call a set W = $\{v_1, v_2, v_1', v_2', \dots v_{n-1}, v_{n-1}', v_n, v_n'\}$.

Theorem 23

For the n-ladder graph $P_2 \times P_n$, $\gamma_{epd}(S(P_2 \times P_n)) = \begin{cases} n-1, & \text{when } S = \{v \in W\}; \\ n-2, & \text{otherwise.} \end{cases}$

Proof.

Let $G = P_2 \times P_n$ be the given n −ladder graph. Note that $|V(G)| = 2n$. Obtain the subdivision of $P_2 \times P_n$ with $V(S(P_2 \times P_n)) = \{v_1, v_2, \dots, v_n, v_1', v_2', \dots v_{n-1}', v_n, v_n'\} \cup \{u_1, u_2, \dots, u_{n-1}, v_1', v_2', \dots, v_{n-1}', w_1, w_2, \dots, w_n\}$. Now one can see that $|d(u) - d(v)| \leq 1$ for every $u, v \in S(P_2 \times P_n)$. Let S denote the required equitable power dominating set of G. Then the following two cases arise.

Case (i): when $S = \{v; v \in W\}$

Without loss of generality, let S = $\{v_1\}$. It is easy to see that $v_1$equitably power dominates u_1 & w_1 whereas u_1 equitably power dominates v_2 and w_1 equitably power dominates v_1'. Moreover, the vertices v_2 and v_2' have two adjacent vertices that cannot be observed and so one has to include one vertex w_2 to be in S. Proceeding this way we get $\gamma_{epd}(S(P_2 \times P_n)) = n - 1$.

Case (ii): when $S \subseteq V\text{-}W$

Consider S = $\{v_3\}$, then v_3 equitably power dominates $w_1, u_3,$ and u_2. Now there are two non-observed vertices for the already observed vertices $w_1, u_3,$ and u_2. None of them equitably power dominate any other vertices. Hence we must choose all the vertices $w_1, w_2, \dots, w_n$ to be in S. Hence $\gamma_{epd}(S(P_2 \times P_n)) = n - 2$.

Definition 24

"The n −barbell graph is obtained by connecting two copies of a complete graph K_n by a bridge."

Theorem 25 [2]

Let G be a n −barbell graph. Then $\gamma_{epd}(G) = 2$, for $n > 2$.

Theorem 26

Let G be a n −barbell graph. Then $\gamma_{epd}\big(S(G)\big) = 2(m+n)+1$, for $n > 6$.

Proof.

Let G be the given n −barbell graph with $(G) = \{v_1, v_2, \dots, v_n, v_1', v_2', \dots, v_n'\}$. Note that when we perform a subdivision on n −barbell graph, the resultant graphs have no adjacent vertices equitably power dominating any of its neighbors. Therefore we must choose all the vertices to be in S. Hence $\gamma_{epd}\,(S(G)) = 2(m+n)+1$.

3. Applications

The concept of domination helps in computer and in communication to route the information between nodes [6, 7]. Eventually power domination plays a vital role in PMU (Phase Measurement Unit), by minimizing the number of units that are placed in the nodes. It is desirable to minimize the PMU and at the same time it monitors the entire system. Circuits with high voltage may get damaged when it is get connected with very low voltage, for smooth conduct of entire system nodes with equal or tend to be equal may have better transmission. We believe that the concept of equitable power domination would play a vital impact in the field of electric power companies and ad-hoc networking.

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