Theoretical Investigation On Mathematical Modelling On Population Growth

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Abstract

This study is aiming to develop a mathematical approach to predict the population growth using different mathematical models. Most recent use of population growth curves has focused on difference equation models (also called "finite population models"). Such models may give a somewhat wider scope for applications and for theory, than do the integrated versions of these models. However, the available integrated versions appear to give much better fits to actual growth curve data, raising some questions about the practical utility of the difference equation versions. We thus examine a number of integrated and difference equation models in this paper.

Keywords: Population growth, Logistic growth model, Carrying capacity, The Malthus's Population Model.

Introduction: Growth in population has become one of the most important and prominent issues in the world. As size and growth of population in any country directly influence the education, economy, culture and environment of that country so population growth plays a vital role as a rational basis for decision making. As gaining accurate information about the future population size supports planning activities. Every government and collective sectors always require accurate idea about the future size of various entities like population, resources, demands and consumptions for their planning activities [1]. To obtain this information, the behaviour of the connected variables is analysed based on the previous data by the statisticians and mathematicians at first, and using the conclusions drawn from the analysis, they make future projections of the aimed at variable. With the population explosion, peoples will have to face the shortage of land, mineral, energy, food, health care, education and so on various aspects resources. This will seriously affect the economic construction, social development and Improving people's lives. Therefore, strictly control the population growth is an important and urgent task for us at present. National population growth, for example, can be used to plan for future Social Security and Medicare obligations ,State growth can be used to determine future water demands, Local growth can be used to determine the need for new public schools and to select sites for fire .Population projections can be used to forecast the demand for housing ,the number of people with disabilities and the number of sentenced criminals. It is population growth that intensifies all these problems.

Mathematical modelling is a broad interdisciplinary science that uses mathematical and computational techniques to model and elucidate the phenomena arising in life sciences. Thus, it is a process of mimicking reality by using the language of mathematics. A mathematical model is defined as "a collection of equations based on quantitative description of a real world phenomenon, it is created in the hope that the predicted behaviour will resemble the actual one" [2].

Many people examine population growth through observation, experimentation or through mathematical modelling. Mathematical models can take many forms, including but not limited to dynamical systems, statistical models and differential equations. The well-known models in population growth estimation are the simple exponential growth and Verhulst Logistic growth Models. The simple exponential growth model can provide an adequate representation to population growth in ideal environment with unlimited resources. While in nature, as population size increase, population growth rate gradually decreases due to limiting factors. While, Population growth rate slows and eventually stops. This is known as Verhulst logistic growth [3] [4]. The population size at which growth stops is generally called the carrying capacity (K), which is the upper bound number of individuals of a particular population that the environment can support [3]. In an environment that will support a limited population it is assumed that the rate of growth of population decreases as the limiting population is approached. An appropriate model (the Verhulst-logistic model) is given by

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{N_m}\right)$$

N_m is the maximum population which can be sustained and r is the intrinsic growth rate.

We can solve this equation for N, assuming $N = N_0$ at $t = 0$.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{N_m}\right)$$

$$\text{Let } x = \frac{N}{N_m} \qquad \text{ie. } \mathrm{N} = \mathrm{N_m} x$$

$$N_m \frac{dx}{dt} = rN_m x(1 - x)$$

$$\frac{\mathrm{dx}}{\mathrm{dt}} = rx(1 - x)$$

$$\frac{dx}{x(1 - x)} = rdt$$

$$\text{that is } \left[\frac{1}{\mathrm{x}} + \frac{1}{1 - x}\right] dx = rdt$$

Integrating both sides we get

$$\ln\left|\frac{\mathrm{x}}{1 - \mathrm{x}}\right| = rt + c$$

Applying boundary conditions we have

$$t = 0, \quad x = x_0 \quad \left(= \frac{N_0}{N_m} \right)$$

$$\therefore \quad \ln\left| \frac{x}{1-x} \cdot \frac{1-x_0}{x_0} \right| = rt$$

taking natural logarithms of both sides

$$\frac{x}{1-x} \cdot \frac{1-x_0}{x_0} = e^{rt}$$

$$x\left(\frac{1}{x_0 - 1} \right) = (1-x)e^{rt}$$

$$\frac{x}{x_0} - x = (1-x)e^{rt}$$

$$0 = \frac{x}{x_0} - (1-x)e^{rt} - x$$

$$0 = x\left(\frac{1}{x_0} + e^{rt} - 1 \right) - e^{rt}$$

$$\therefore x = \frac{e^{rt}}{\left(\frac{1}{x_0} + e^{rt} - 1 \right)}$$

to tidy up expression multiply both sides by e^{-rt}

$$e^{-rt} x = \frac{1}{\left(\frac{1}{x_0} + e^{rt} - 1 \right)}$$

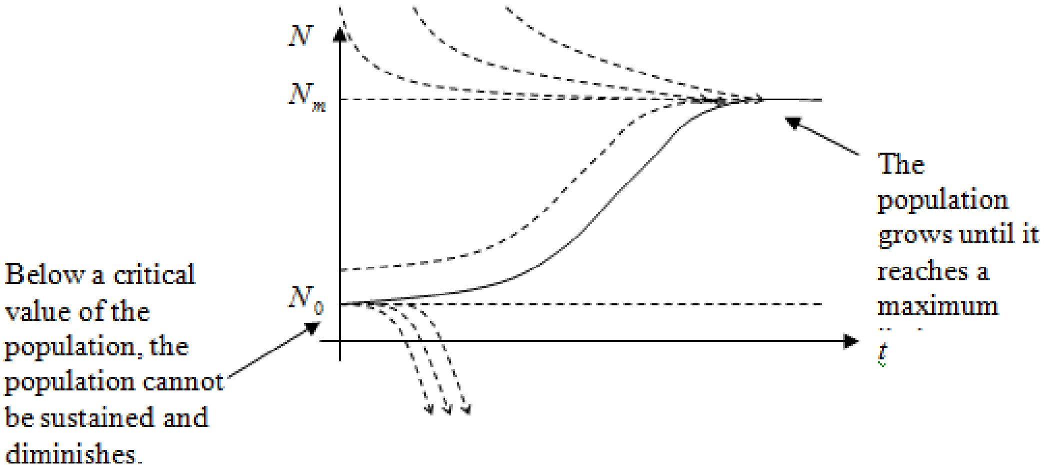
$$x = \frac{1}{1 + \left(\frac{1}{x_0} - 1 \right) e^{-rt}}$$

Now putting $x = \frac{N}{N_m}$ and $x_0 = \frac{N_0}{N_m}$ back into the above expression, we get

$$\frac{N}{N_m} = \frac{1}{1 + \left(\frac{N_m}{N_0} - 1 \right) e^{-rt}}$$

$$N = \frac{N_m}{1 + \left(\frac{N_m}{N_0} - 1 \right) e^{-rt}}$$

A sketch can illustrate this result.



Statistical Software packages are used to build a statistical model for the population estimation is becoming more widely raises the value of interest [5]. Many websites such as [5] gave predictions for the population of different countries or regions. However, in many cases these estimations are rough and can be risky to be used in planning activities especially by governments. This work proposes two different approaches to predict the population of a country or region based on current available data. A statistical Software package is used to build a statistical model for the population estimation rate. Studying population growth is becoming more widely raises the value of interest [5]. In my work, I am going to review the work done by people on Population growth using various mathematical models.

Review of Literatures

In order to study how the world population changes over time, it is useful to consider the *rate* of change rather than focusing only on the total population level. The following visualization presents the annual population growth rate superimposed over the total world population for the period 1750-2010, as well as projections up to 2100 . This is the period in history when population growth changed most drastically. Before 1800, the world population growth rate was always well below 1%. Over the course of the first fifty years of the 20th century, however, annual growth increased to up to 2.1%—the highest annual growth rate in history, which was recorded in 1962. Since peaking, the growth rate has systematically been going down, with projections estimating an annual rate of 0.1% for 2100. This means that while the world population quadrupled in the 20th century, it will not double in the 21st century [5].

Wali et al [6] has observed the population growth of Rwanda. Rwanda is a small landlocked African country located in Central Africa. It borders Democratic Republic of Congo, Uganda, Tanzania and Burundi. It has a total area of 26,338 square kilometers comprising 24,668 square kilometers of land and 1,670 square kilometers of water. Most of the country is savanna grassland with the population predominantly rural. It has a very high population growth.Ideally if the population continues to grow without bound, nature will take over

and the death rate will rise to solve the problem. Unfortunately, this is not the most attractive scenario: instead the birth rate would rather be controlled in order to reduce population growth. In relation to his work, secondary classified yearly population data of Rwanda from 1980-2008 (inclusive) were collected from International Data Base (IDB) and National Institute of Statistics of Rwanda (NISR). The Logistic growth mathematical model was used to compute predicted population values employing MATLAB.

The Statistical Package for Social Sciences (SPSS) was also used to plot the graph of the classified yearly population data of Rwanda from 1980 to 2008 (inclusive). In 1798 the Englishman Thomas R. Malthus [7], proposed a mathematical model of population growth. The differential equation governing population growth in this case is

$$\frac{d}{dt}N(t) = aN(t)$$

where, t represents the time period and a, referred to as the Malthusian factor, is the multiple that determines the growth rate. This equation is a non-homogeneous linear first order differential equation known as Malthusian law of population growth.

Wali et al. used this model to determine the carrying capacity and the vital coefficients governing the population growth of Rwanda. He modified Malthus's Model to make the population size proportional to both the previous population and a new term

$$\frac{a - bN(t)}{a}$$

where a and b are called the vital coefficients of the population. This term reflects how far the population is from its maximum limit. However, as the population value grows and gets closer to a/b , this new term will become very small and get to zero, providing the right feedback to limit the population growth. The ratio a/b is called the carrying capacity of population and in case of Rwanda the predicted carrying capacity for the population is 77208025.64. In the case of Rwanda it was found out that the vital coefficients a and b are 0.03 and 3.885606419 × 10−10 respectively. Thus the population growth rate of Rwanda, according to this model, is 3% per annum.

Li et al. [8] applied improved Logistic model which are called Logarithm Logistic Models for perdiction of population growth. He compared different models such as such as Malthus population model (also called be exponential growth model), the Logistic growth model and power law exponent model, etc. Each model has its advantages and shortages [9].Malthus, a famous British demographer, put forward the assumption that the population growth rate is constant by analyzing the British population survey data. Then, he gives the following exponential growth model.

$$x(t) = x_0 e^{rt}$$

where r>0 is the unchanged growth rate and x0 is the initial population.Malthus model obtains a good fitting result in the early,but the result is bad in the late. An important reason is the limit of exponential growth model will tend to infinity. So, this model is only suitable for description the growth of a biological population at the initial stage. The main error of Malthus model is the unchanged growth rate. We know that the growth rate of population is closely related to many factors, such as the population migration, the restrictions of living space, available water, infectious diseases, birth rate and mortality rate, and the mortality rate is affected by public health conditions, war, pollution, the level of raw materials, eating habits and psychological pressure. To correct this error, in 1840s, Pierre-Francois Verhulst [10, 11], a bio-mathematician of Denmark, gave an improved growth model by analyzing the effect of natural resources, environmental conditions and other factors. He thinks the natural resources and environmental conditions will block the growth of population. Verhulst assume that the growth rate is a linear function, i.e. $r(x) = r - sx$, $(s>0, r>0)$ where r is the unchanged growth rate at the initial stage. Let $x=x_m$ where x_m is the maximum population capacity, it means that

$$r(x_m) = 0$$

then we have $s=r/x_m$. Therefore, we obtain the traditional Logistic model as follows.

$$\begin{cases} \frac{dx}{dt} = r\left(1 - \frac{x}{x_m}\right)x \\ x(0) = x_0 \end{cases}.$$

Its solution is the following function, which is called S curve.

$$x(t) = \frac{x_m}{1 + \left(\frac{x_m}{x_0} - 1\right)e^{-rt}}.$$

The Logistic model is widely used in many fields, such as yeast number of culture in limited space, output prediction of new product in economy and future population forecasts. Some experiments show that the Logistic model is close to the actual value. Different from the Logistic model, the authors in [12] assume that the growth rate is power law exponent, i.e.

$$r(x) = k(x - x_m)^{\alpha}$$

They point out the power law index of the model has important affect on the total number of some particular biological populations. In general, the Logistic model is a good prediction tool.But the hypothesis of linear function is irrational in some conditions. Actually, the growth rate will decline with the increasing of population, furthermore, the rate of decline is getting slower. Therefore, Li et al. assume that

$$r(x) = r\left(1 - log_{x_m} x\right)^k.$$

The following Figure 1 is the curve of the growth rate under the condition k=1, r=1 and x_m= 350.

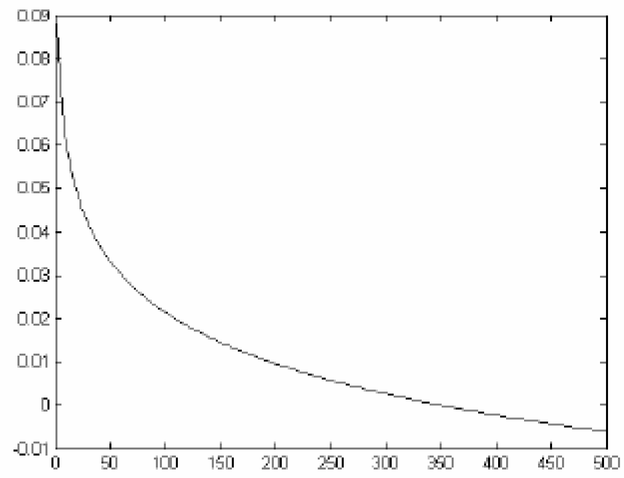


Figure 1. The curve of growth rate

According to the basic population equation, Auther [8] improved the traditional growth model, and get the following differential equation model:

$$\begin{cases} \frac{dx}{dt} = r\left(1\text{-}log_{x_m} x\right)^k x \\ x(0) = x_0 \end{cases},$$

Where x_m is the maximum population capacity and r is the unchanged growth rate at the initial stage.It is called be improved logarithm Logistic model.

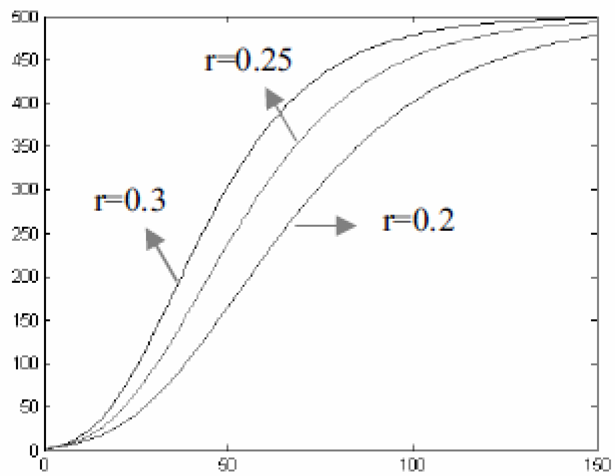


Figure 2. The logarithm Logistic curve of model (3) under different parameters

Based on U.S.Census data, the parameters of the models were estimated by applying the Least Squares Method. The experiments show that the prediction value of the new models is much closer to the actual value than the classical Logistic model. Finally, through analyzing the rationality of the maximum population capacity, the trend of Logistic curves and the rationality of prediction value, the most appropriate model is recommended .

Matintu [13] has also used mathematical models to predict the population growth of Ethiopia in 2016. Ethiopia is an overpopulated country in Africa next to Nigeria. It shares a border with Eritrea to the North and Northeast,Djibouti and Somalia to the East, Sudan and South Sudan to the West, and Kenya to the South. The Malthus's and the logistic growth models are applied to model the population growth of Ethiopia using data from to .The data used was collected from International Data Base (IDB). We also use the least square method to compute the best population growth rate of Malthus's model. The Malthus's population model predicted a growth rate of per year while the logistic growth model predicted the carrying capacity of and growth rate of per annum. The growth rate for both models match well with the growth rate estimated by International Data Base for the past four years. The Mean Absolute Percentage Error is computed as for Malthus's population model and for logistic growth Model. This showed that the Malthus's population model seems to fit the original data the best among the models we tried.

M.Zabadi et al. [14]has done study to develop a mathematical approach to predict the population of Jordan until year 2100. This approach applies the simple exponential growth model and the Verhulst logistic growth equation to predict the population of Jordan utilizing predated data from 1955 to 2016. The explicit solutions for each model are exactly derived by using mathematical techniques of differentiation and integration. A Non-Linear Regression analysis was applied through Minitab. The curve-fitting tool (cftool) of MATLAB was used. Results show that the exponential model predicted the population of Jordan to be 123.169 Million in 2100, with a growth rate of 3.27% per annum. The Logistic model predicted Jordan's population to be 17.346 Million in 2100, with growth rate of 4.56 %. While, the Verhulst growth equation predicted the population of Jordan to be 12.157 Million in 2100, with a growth rate of 5.25%.

Conclusion: A comparison between outputs of the three models was conducted to reveal that exponential model cannot be used, Logistic Based Models are more reasonable. In this paper, we first assume that the growth rate is gradually decline with the increasing population, and then two improved Logarithm Logistic models are given. Furthermore, the analytical solutions of the models are obtained. Finally, these models are applied to predict the population. The experiments show that our new prediction models are better than the traditional Logistic model in some cases The models can be not only applied to predict the population, but also applied to describe the growth of the other biological populations.

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