SOME APPLICATIONS OF MEASURES ON THE CARTESIAN PRODUCT

D. R. Patel[1*], H. S. Mehta[2] and R. D. Mehta[3]
[1]Department of Mathematics, BVM engineering college, Vallabh Vidyanagar, India
[2,3]Department of Mathenmatics, Sardar Patel University, Vallabh Vidyanagar, India

Abstract: *It is well known that measures play an important role in the study of function algebras, e.g., representing measures, annihilating measures etc. We have studied annihilating measures for the Cartesian product of two function algebras in detail. Here we discuss representing measures for the Cartesian product. We also discuss some applications of these two measures on the Cartesian product.*

Keywords: Function algebra, Cartesian product, Representing measure, Annihilating measure.

1. INTRODUCTION

Let A and B be function algebras on compact Hausdorff spaces X and Y respectively. Then $A\times B$ is a function algebra on X + Y with respect to the norm $\|(f,g)\| = \max\{\|f\|_{\infty,} \|g\|_{\infty}\}$ [4]. In general, $(A\times B)^* = A^*\times B^*$, for any two Banach spaces A and B, where the norm on $A\times B$ is the maximum norm (as above), the norm on $A^*\times B^*$ is $\|\phi\| = \|(\phi_{A,}\phi_B)\| = \|\phi_A\| + \|\phi_B\|$ and A^* is the dual of A. As a consequence we get $M(X+Y)\cong M(X)\times M(Y)$, where M(X) denotes the Banach space of all regular complex Borel measures on X, because $C(X+Y)\cong C(X)\times C(Y)$ and $C(X)^* = M(X)$. The association of measures is given by $\eta(E\cup F) = \mu(E)+\nu(F)$, where $\mu\in M(X)$ and $\nu\in M(Y)(E\subset X, F\subset Y)$.

Definition 1.1. [3] A measure $\mu\in M(X)$ is said to be global if supp μ = X, where supp μ denotes the support of a measure μ. It is easy to check the following.

Proposition 1.2. Let $\eta = (\mu,\nu)\in M(X+Y)$. Then

(1) supp η = supp μ $\cup$ supp ν.

(2) η is a global measure if and only if μ and ν are.

Definition 1.3. [1] Let A be function algebra on X. A measure $\mu\in M(X)$ is said to be an annihilating measure for A, if $\int_X f d\mu = 0, \forall f\in A$. The set of all annihilating measures for A is denoted by $A^\perp$. The characterization of $(A\times B)^\perp$ is discussed in detail in [5]. The result that we need is as follows.

Proposition 1.4. Let A and B be function algebras on X and Y respectively. Then $(A\times B)^\perp = A^\perp\times B^\perp$.

2. Representing measures

In this section, we relate the representing measures for function algebras A, B with that of $A\times B$. Let $\Delta(A)$ denotes the maximal ideal space of A. Note that $\Delta(A\times B) = \Delta(A)+\Delta(B)$ with the Gelfand topology [6]. For if $\phi\in\Delta(A\times B)$, then $\phi = \phi_A\in\Delta(A)$ or $\phi = \phi_B\in\Delta(B)$, where ϕ_A and ϕ_B are defined as $\phi_A(f) = \phi_A(f,0), \forall f\in A$ and $\phi_B(g) = \phi_B(0,g), \forall g\in B$. On the other hand, If $\phi_A\in\Delta(A)(\phi_B\in\Delta(B))$, then $\phi\in\Delta(A\times B)$ is defined as $\phi(f,g) = \phi_A(f), (\phi(f,g) = \phi_B(g)), \forall (f,g)\in A\times B$.

Definition 2.1. [1] Let A be a function algebra on X and $\phi\in\Delta(A)$. A representing measure for ϕ is a positive measure μ on X such that $\phi(f) = \int_X f d\mu, \forall f\in A$.

Theorem 2.2. Let $\eta=(\mu,\nu)$ be a representing measure for $\phi\in\Delta(A\times B)$ on X + Y. Then μ is a representing measure for ϕ_A, if $\phi=\phi_A\in\Delta(A)$ and ν is a representing measure for ϕ_B, if $\phi=\phi_B\in\Delta(B)$. On the other hand, if $\mu\in M(X)(\nu\in M(Y))$ is a representing measure for $\phi_A\in\Delta(A)$ $(\phi_B\in\Delta(B))$, then $\eta=(\mu,0)(\eta=(0,\nu))$ is a representing measure for $\phi=\phi_A\in\Delta(A\times B)(\phi=\phi_B\in\Delta(A\times B))$

Proof. Let $\phi=\phi_A\in\Delta(A)$. Since η is a representing measure for ϕ, we have $\phi(f,g)=\int_{X+Y}(f,g)d\eta,\forall(f,g)\in A\times B$. But $\eta=(\mu,\nu)$ with $\mu\in M(X)$ and $\nu\in M(Y)$ and as $\phi=\phi_A\in\Delta(A),\phi_B=0$ [6] and so $\phi_A(f)=\phi(f,0)$. Therefore $\phi_A(f)=\int_{X+Y}(f,0)d(\mu,\nu)=\int_X fd\mu+\int_X 0d\nu=\int_X fd\mu,\forall f\in A$. Also since η is positive, μ is positive. Therefore μ is a representing measure for ϕ_A. Similarly, if $\phi=\phi_B\in\Delta(B)$, then ν is a representing measure for ϕ_B.

Conversely, assume that μ is a representing measure for $\phi_A\in\Delta(A)$. Then $\phi_A(f)=\int_X fd\mu,\forall f\in A$ and since $\mu\in M(X)$ is positive, $\eta=(\mu,0)\in M(X+Y)$ is also positive. Note that $\phi=\phi_A\in\Delta(A\times B)$ and so ϕ_B = 0. Now let $h=(f,g)\in A\times B$. Then $\phi(h)=\phi_A(f)+\phi_B(g)=\phi_A(f)=\int_{X+Y}(f,g)d(\mu,0)=\int_{X+Y}hd\eta$. Thus $\phi(h)=\int_{X+Y}hd\eta,\forall h\in A\times B$. Hence η is a representing measure for $\phi\in\Delta(A\times B)$. Similarly, if ν is a representing measure for $\phi=\phi_B\in\Delta(A\times B)$, then $(0,\nu)$ is a representing measure for $\phi=\phi_B\in\Delta(A\times B)$.

Definition 2.3. [1] The Choquet boundary of A, denoted by *ch* (A), is the set of all $x\in X$ such that ϕ_x admits a unique representing measure, where ϕ_x is the point evaluation at x.

Corollary 2.4. $ch(A\times B)$ = *ch* (A) + *ch* (B).

There are some special types of representing measures associated with function algebra.

Definitions 2.5. [1] Let A be a function algebra on X and $\phi\in\Delta(A)$.

(1) A Jensen measure for ϕ is a positive measure μ in M(X) such that $\log|\phi(f)|\le\int_X\log|f|d\mu,\forall f\in A$.

(2) An Arens-Singer measure for ϕ is a positive measure μ in M(X) such that $\log|\phi(f)|=\int_X\log|f|d\mu,\forall f\in A^{-1}$, where A^{-1} is the set of all invertible elements in A.

Theorem 2.6. If μ is a Jensen measure (Arens-Singer measure) for $\phi_A\in\Delta(A)$, then $\eta=(\mu,0)$ is a Jensen measure (Arens-Singer measure) for $\phi=\phi_A\in\Delta(A\times B)$. Also if ν is a Jensen measure (Arens-Singer measure) for $\phi_B\in\Delta(B)$, then $\eta=(0,\nu)$ is a Jensen measure (Arens-Singer measure) for $\phi=\phi_B\in\Delta(A\times B)$.

On the other hand, if $\eta=(\mu,\nu)$ is a Jensen measure (Arens-Singer measure) for $\phi\in\Delta(A\times B)$, then μ is a Jensen measure (Arens-Singer measure) for ϕ_A, if $\phi=\phi_A\in\Delta(A\times B)$ and ν is a Jensen measure (Arens-Singer measure) for ϕ_B, if $\phi=\phi_B\in\Delta(B)$

Proof. Suppose μ is a Jensen measure for $\phi_A\in\Delta(A)$. Then $\log|\phi(f)|\le\int_X\log|f|d\mu,\forall f\in A$ and since $\mu\in M(X)$ is positive, $\eta=(\mu,0)\in M(X+Y)$ is positive. Also $\phi=\phi_A\in\Delta(A\times B)$. Now for $h=(f,g)\in A\times B$, $\log|\phi(h)|=\log|\phi_A(f)|\le\int_X\log|f|d\mu$. Thus $\log|\phi(h)|\le\int_{X+Y}\log|h|d\eta,\forall h\in A\times B$. Therefore η is a Jensen measure for $\phi\in\Delta(A\times B)$.

Using similar arguments as above and the fact that $(A\times B)^{-1}=A^{-1}\times B^{-1}$ [6], one can show that $\eta=(\mu,0)$ is an Arens-Singer measure for $\phi=\phi_A\in\Delta(A\times B)$, whenever μ is an Arens-Singer measure for $\phi_A\in\Delta(A)$.

Similarly, if ν is a Jensen measure (Arens-Singer measure) for $\phi_B \in \Delta(B)$, then $\eta=(0,\nu)$ is a Jensen measure (Arens-Singer measure) for $\phi=\phi_B \in \Delta(A\times B)$.

Conversely, assume that η is a Jensen measure for $\phi\in\Delta(A\times B)$. Then $\log|\phi(h)|\le \int_{X+Y}\log|h|\,d\eta, \forall h=(f,g)\in A\times B$ and $\eta=(\mu,\nu)\in M(X+Y)$ with $\mu\in M(X)$ and $\nu\in M(Y)$. Further, since μ is positive, μ and ν are also positive. Since $\phi\in\Delta(A\times B)$, either $\phi\in\Delta(A)$ or $\phi\in\Delta(B)$. Suppose $\phi\in\Delta(A)$. Then $\phi=\phi_A$. Now for $f\in A$, $\log|\phi_A(f)|=\log|\phi(f,0)|\le\int_X\log|f|\,d\mu$. Thus $\log|\phi_A(f)|\le\int_X\log|f|\,d\mu, \forall f\in A$. Therefore μ is a Jensen measure for $\phi=\phi_A\in\Delta(A)$. If η is an Arens-Singer measure for $\phi=\phi_A\in\Delta(A\times B)$, then by taking the function $h=(f,1)\in(A\times B)^{-1}$ and using similar arguments as above, one can show that μ is an Arens-Singer measure for ϕ_A, if $\phi_A\in\Delta(A)$. Similarly, if $\phi\in\Delta(B)$, then ν is a Jensen measure (Arens-Singer measure) for $\phi=\phi_B\in\Delta(B)$.

3. Applications of measures

There is a hierarchy of function algebras such as Dirichlet algebras, logmodular algebras, URM algebras etc. which are related with representing and annihilating measures. Here we discuss their Cartesian product. Since p-set is defined in terms of annihilating measures, we also discuss it for the Cartesian product. Recall that a p-set for a function algebra A on X is a closed subset E of X such that $\mu_E\in A^\perp$, whenever $\mu\in A^\perp$. Here μ_E is the restriction of μ to E [2].

Proposition 3.1. A p-set for A (for B) is a p-set for $A\times B$. On the other hand, if $K\subset X+Y$ is a p-set for $A\times B$, then $K\cap X$ and $K\cap Y$ are p-sets for A and B respectively.

Proof. Let $K\subset X$ be a p-set for A and $\eta=(\mu,\nu)\in(A\times B)^\perp$. Then $\mu\in A^\perp$ and $\nu\in B^\perp$, by Theorem 1.4. So $\mu_K\in A^\perp$. Hence $\eta_K=(\mu,\nu)_{|K}=(\mu_K,0)\in(A\times B)^\perp$. Therefore K is a p-set for $A\times B$. Similarly, a p-set for B is also a p-set for $A\times B$.

Conversely, let $K\subset X+Y$ be a p-set for $A\times B$ and $\mu\in A^\perp$. Then by Theorem 1.4, $\eta=(\mu,0)\in(A\times B)^\perp$ and so $\eta_K\in(A\times B)^\perp$. Note that $\eta_K=(\mu_{K\cap X},0)$ and hence $\mu_{K\cap X}\in A^\perp$. Therefore $K\cap X$ is a p-set for A. Similarly, $K\cap Y$ is a p-set for B.

Remark 3.2. The above proposition can be proved for peak sets and weak peak sets directly also. For function algebras weak peak sets and p-sets are the same.

Definitions 3.3. [1] A function algebra A on X is said to be

(1) a Dirichlet algebra, if Re(A) is dense in $C_R(X)$.

(2) a logmodular algebra, if $\log|A^{-1}|=\{\log|f|: f\in A^{-1}\}$ is dense in $C_R(X)$.

Proposition 3.4. (1) $A\times B$ is a Dirichlet algebra if and only if A and B are Dirichlet algebras.

(2) $A\times B$ is a logmodular algebra if and only if A and B are logmodular algebras.

Proof. It can be easily verified that $\mathrm{Re}(A\times B)=\mathrm{Re}(A)\times\mathrm{Re}(B)$ and $\log|(A\times B)^{-1}|=\log|A^{-1}|\times\log|B^{-1}|$. Hence the result follows.

Remark 3.5. Proposition 3.4 (1) can also be proved using the characterization of Dirichlet algebras, viz. ` A function algebra A is a Dirichlet algebra if and only if the only real annihilating measure of A is zero.'

Definition 3.6. [1] A function algebra A on X is said to be a URM algebra, if every $\phi\in\Delta(A)$ has a unique representing measure.

Proposition 3.7. $A\times B$ is a URM algebra if and only if A and B are URM algebras.

Proof. It follows from Theorem 2.2 and the fact that $\Delta(A\times B)=\Delta(A)+\Delta(B)$.

Proposition 3.8. [3] A function algebra A is pervasive if and only if every nonzero annihilating measure of A is global.

There is a class of algebras intermediate to pervasive algebras and antisymmetric algebras, as follows.

Definitions 3.9. Let A be a function algebra on X. Then

(1) A is said to have global measures if every $\phi\in\Delta(A)-X$ has a representing measure which is global.

(2) A is said to be a maximum modulus algebra, if $X \neq \Delta(A)$ and if $f \in A$ and $\hat{f}$ assumes its maximum modulus at a point of $\Delta(A) - X$, then f is constant.

The function algebra C(X) has no global measures and the disk algebra on the circle, $A(\Gamma)$ has global measures.

Let A be a function algebra on X. Then

A has global measures and $X \neq \Delta(A) \Rightarrow A$ is a maximum modulus algebra $\Rightarrow$ A is antisymmetric [3].

Remarks 3.10. (1) Since $A \times B$ can never be an antisymmetric algebra, even if A and B are [4], $A \times B$ can't be a maximum modulus algebra even if A and B are. Also $A \times B$ can't have any global measure, even if A and B have (assuming $X \neq \Delta(A), Y \neq \Delta(B)$).

(2) Note that every proper pervasive function algebra is analytic and every analytic function algebra is antisymmetric. Hence $A \times B$ can't be pervasive or analytic even if A and B are.

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