Spline Approximation of Velocity Profile For Fluid Flow in Two Dimensions

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Abstract: *In this article, we study the incompressible potential fluid flow in two dimensions. It is assumed that the flow is irrotational and inviscid also. The governing non-linear ordinary differential equation is solved numerically using quartic spline collocation method as well as analytically using Maclaurin's series. The solutions of the problem are obtained in the form of velocity profile. Comparison of the results is shown graphically.*

Keywords: *Quartic spline collocation method, potential flow, Stream function, Maclaurin's Series, Quasilinearization.*

INTRODUCTION

The theory of potential flow of a viscous fluid was introduced by Stokes [1]. All of his work on this topic is framed in terms of the effects of viscosity on the attenuation of small amplitude waves on a liquid–gas surface. Potential flow has been analyzed by many authors[2-5]. In this paper, for the potential flow, we assume that:

(a) Motion is irrotational, so no vortices occur.

(b) Fluid is incompressible, that means its density is uniform.

(c) Fluid is inviscid, so there is no loss, and the fluid does not cross a streamline, but follows along it.

A streamline is a curve whose tangent indicates the direction of the resultant velocity of flow of fluid at the point.

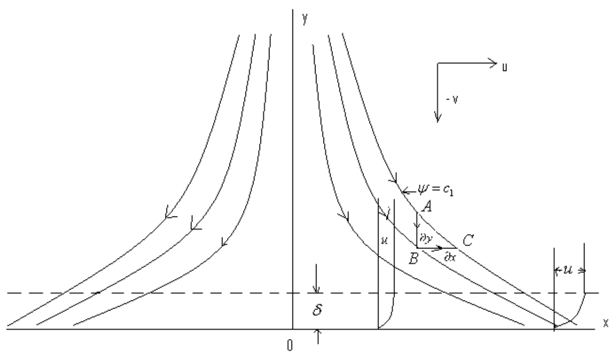

Fig. 1: Diagram Illustrating Streamlines in Flow in Two Dimensions

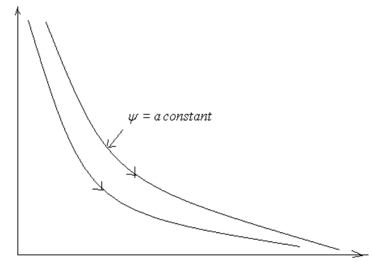

Fig. 2:*Diagram Illustrating Streamlines Fluid inFluid Flow in Region $x \ge 0, y \ge 0$*

The stream function $\psi(x, y)$ symbolically represents the flux or volume of fluid crossing a curved surface whose base line is in the x-y plane, of unit height perpendicular to that plane, in unit time as shown in figures 1 and 2. The streamlines are given by $\psi = a$ constant. Due to the irrotational flow, there is a velocity potential ϕ whose gradients with respect to x and y gives the velocities u, υ in these directions. The resultant velocity along the stream line is $(u^2 + \upsilon^2)^{1/2}$.

Then $$u = \frac{\partial \phi}{\partial x} \quad and \quad \upsilon = \frac{\partial \phi}{\partial y} \qquad (1)$$

Referring to fig.1, u is the velocity of flow perpendicular to $AB = \partial y$, the flux across ∂y is $\partial \psi = u \partial y$, therefore

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \qquad (2)$$

Also (- υ) is the velocity of flow perpendicular to $BC = \partial x$, so the flux across it is $\partial \Psi = -\upsilon \partial x$, therefore

$$\upsilon = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \qquad (3)$$

From equations (2) and (3),

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad and \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad (4)$$

These are the Cauchy-Riemann conditions for the existence of a 'flow' function such that

$$f(z) = f(x+iy) = \phi(x, y) + i\psi(x, y) \qquad (5)$$

By virtue of equation (4), ϕ *and* ψ are conjugate functions and satisfy Laplace's equation in two dimensions,

$$\text{i.e.}\ \nabla^2_{x,y}\phi = \nabla^2_{x,y}\psi = 0$$

Moreover, the streamlines $\psi = c_1$, and the equipotential lines $\phi = c_2$ intersect orthogonally.

Determination of Potential and Stream Functions in Plane Flow:

Let the flow in the x-direction have a constant velocity a as displayed in figure 3, then by equation (2),

$$\frac{\partial \psi_1}{\partial y_1} = a, \quad so \quad \psi_1 = ay_1, \qquad (6)$$

and $\dfrac{\partial \phi_1}{\partial x_1} = a, \quad so \quad \phi_1 = ax_1, \qquad (7)$

The constants of integration being zero, since

$$\left.\begin{matrix}\psi_1\\ \phi_1\end{matrix}\right\} = 0 \quad when \quad \left.\begin{matrix}y_1\\ x_1\end{matrix}\right\} = 0$$

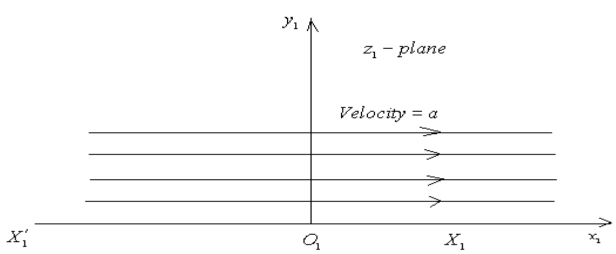

Fig.3 : *Diagram Illustrating Streamlines in Fluid Flow Parallel to X_1 – axis, in Region*

$-\infty < x_1 < \infty, \quad y_1 \geq 0$

Thus the flow function is

$$f(z_1) = \phi_1(x, y) + i\psi(x, y) = az_1 \qquad (8)$$

To determine the flow in the region $y \geq 0, \ x \geq 0,$ we use a conformal transformation such that the part of the $x_1 - axis$ $O_1X_1^{'}$ in figure 3, becomes the y-axis in figure (2). Accordingly we put $z = z_1^{1/2}$ *or* $z^2 = z_1$, which gives

$$az_1 = a\left(x^2 - y^2\right) + 2iaxy, \qquad (9)$$

so the new potential and stream functions are, respectively,

$$\phi = a\left(x^2 - y^2\right), \tag{10}$$

and $$\psi = 2axy \tag{11}$$

The streamlines $\psi = c$ are the rectangular hyperbolae

$$xy = \frac{c}{2a}, \quad a \text{ is constant} \tag{12}$$

while the equipotential lines, orthogonal to the streamlines are those parts of the rectangular hyperbolae

$$x^2 - y^2 = b^2, \text{ a constant, or } \frac{x^2}{b^2} - \frac{y^2}{b^2} = 1, \tag{13}$$

whose common asymptote is the straight line $y = x$ which is seen in figure 4.

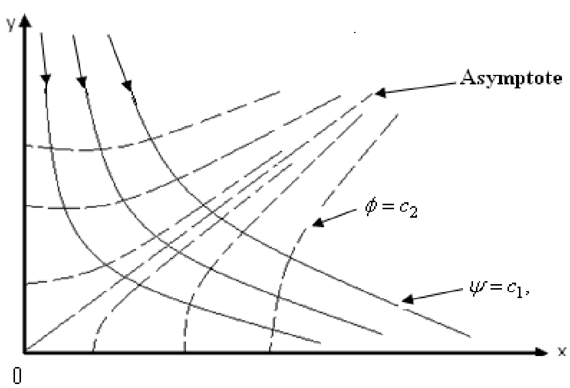

Fig.4 : *Equipotential Lines Orthogonal to the Streamlines Inclusion of Small Viscosity*

In absence of viscosity, by equation (11), the stream function is

$$\psi(x, y) = 2axy. \tag{14}$$

When small viscosity is included, we write

$$\psi(x, y) = x\,g(y), \tag{15}$$

where $g(y)$ is a function of y to be determined. From (2), (3) and (15),

$$u = xg'(y) \;\; and \;\; -\upsilon = g(y) \tag{16}$$

When y is positive and large enough, the influence of viscosity is assumed to be negligible, and then ψ is given by equation (14). Thus as $y \to \infty$, equation (14) together with (2) and (3) yields the boundary conditions

$$u = 2ax, \quad \upsilon = -2ay. \tag{17}$$

According to (16), (17) for $x > 0$

$$g = 2ay, \quad g' = 2a, \quad y \to \infty; \tag{18}$$

while at the surface of the plate where $y = 0$, we have $u = \upsilon = 0$, so

$$g(0) = g'(0) = 0 \tag{19}$$

Using the transformation $z^2 = z_1$, *for* $u, \upsilon,$ to have dimensions lt^{-1}, that of a must now bet^{-1}.

Governing Differential Equation for *g*(*y*):

In absence of viscosity, the pressure is given by Bernoulli's equation

$$p = p - \frac{1}{2}\rho \upsilon el^2$$

$$\therefore\ p = p_o - 2\rho a^2\left(x^2 + y^2\right), \tag{20}$$

the velocity being $2a\left(x^2 + y^2\right)^{1/2}$. For viscous flow we take

$$p = p_0 - 2a^2\rho\left[x^2 + G(y)\right], \tag{21}$$

where $G(y)$ is a function to be determined.

The Navier-Stokes equations for steady fluid flow are

$$u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial \rho}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{22}$$

and $$u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} = -\frac{1}{\rho}\frac{\partial \rho}{\partial y} + \nu\left(\frac{\partial^2 \upsilon}{\partial x^2} + \frac{\partial^2 \upsilon}{\partial y^2}\right), \tag{23}$$

where $\nu = \dfrac{\mu}{\rho}$ is the kinematic viscosity of the fluid.

μ is the dynamic viscosity having dimensions $ml^{-1}t^{-1}$

From the equations (16), (21) and (22), we obtain

$$\nu\frac{d^3 g}{dy^3} + g\frac{d^2 g}{dy^2} - \left(\frac{dg}{dy}\right)^2 + 4a^2 = 0 \tag{24}$$

Writing $\xi = \left(\dfrac{2a}{\nu}\right)^{1/2} y,\ (2a\nu)^{1/2} w(\xi) = g(y),\ gives$

$$\frac{dg}{dy} = \frac{dg}{d\xi}.\frac{d\xi}{dy} = (2a\nu)^{1/2} w'(\xi)\left(\frac{2a}{\nu}\right)^{1/2} = 2aw', \qquad (25)$$

therefore,
$$\frac{d^2g}{dy^2} = 2a\frac{dw'}{d\xi}.\frac{d\xi}{dy} = \left[(2a)^{3/2}/\nu^{1/2}\right]w'', \qquad (26)$$

and
$$\frac{d^3g}{dy^3} = \left[(2a)^{3/2}/\nu^{1/2}\right]\left(\frac{2a}{\nu}\right)^{1/2} w''' = 4a^2\frac{w'''}{\nu} \qquad (27)$$

The use of equations (25) to (27) into (24) leads to the third order nonlinear equation

$$w''' + ww'' - w'^2 + 1 = 0 \qquad (28)$$

associated with the three boundary conditions

$$w(0) = 0,$$
$$w'(0) = 0 \qquad (29)$$
$$w'(\infty) = 1,\ for\ all\ x \geq 0$$

SOLUTION USING MACLAURIN'S SERIES

The Maclaurin's series for the solution to the equation (28) is given by

$$w(\xi) = w_0 + \xi w_0^{(1)} + \frac{\xi^2}{2!}w_0^{(2)} + ... + \frac{\xi^n}{n!}w_0^{(n)} \qquad (30)$$

The boundary conditions at origin are w=0, w'=0, and also from (28), it follows that $w_0^{(3)} = -1$. $w_0^{(2)}$ is unknown, so call it α. By repeating differentiation of (28) and inserting the values of $w_0^{(n)}$ from the Abac into (30), yields the series solution

$$w(\xi) = -\frac{1}{3!}\xi^3 + \frac{2}{7!}\xi^7 + \frac{16}{11!}\xi^{11} + \frac{2128}{15!}\xi^{15} + \frac{721664}{19!}\xi^{19} + ... +$$
$$+\alpha\left\{\frac{1}{2!}\xi^2 - \frac{2}{6!}\xi^6 - \frac{16}{10!}\xi^{10} - \frac{2128}{14!}\xi^{14} - \frac{721664}{18!}\xi^{18}.....\right\} +$$

$$+\alpha^2\left\{\frac{1}{5!}\xi^5+\frac{4}{9!}\xi^9+\frac{840}{13!}\xi^{13}+\frac{299684}{17!}\xi^{17}+\frac{212311872}{21!}\xi^{21}+....\right\}$$

$$-\alpha^3\left\{\frac{1}{8!}\xi^8+\frac{18}{12!}\xi^{12}+\frac{70117}{16!}\xi^{16}+\frac{52214273}{20!}\xi^{20}+....\right\}$$

$$+\alpha^4\left\{\frac{27}{11!}\xi^{11}+\frac{10725}{15!}\xi^{15}+\frac{8391913}{19!}\xi^{19}+....\right\}$$

$$-\alpha^5\left\{\frac{951}{14!}\xi^{14}+\frac{916656}{17!}\xi^{18}....\right\}$$

$$+\alpha^6\left\{\frac{51465}{17!}\xi^{17}+\frac{92852282}{21!}\xi^{21}+....\right\}$$

$$-\alpha^7\left\{\frac{3355837}{20!}\xi^{20}+....\right\}+... \qquad (31)$$

$w'(\xi)$ is the differentiation of (31).

The value of α must be determined to satisfy the boundary condition at infinity, i.e. w'=1 when $\xi=\infty$.

QUARTIC SPLINE COLLOCATION METHOD

The fourth degree spline is used to find numerical solutions to the boundary value problems discussed in equation (28) together with boundary conditions (29). A detailed description of spline functions generated by subdivision is given by De Boor [6], Micula et al [7].

Consider equally spaced knots of a partition π: $a = x_o < x_1 < x_2 < ... < x_n = b$ on [a, b]. Let $S_4[\pi]$ be the space of continuously differentiable, piecewise, Quartic polynomials on π. That is, $S_4[\pi]$ is the space of Quartic polynomials on π. The Quartic spline is given by Bickley [8].

$$s(x) = a_0 + b_0(x-x_0) + \frac{1}{2}c_0(x-x_0)^2 + \frac{1}{6}d_0(x-x_0)^3 + \frac{1}{24}\sum_{k=0}^{n-1} e_k(x-x_k)_+^4 \quad (32)$$

where the power function $(x-x_k)_+$ defined as

$(x-x_k)_+ = x - x_k$, if $x > x_k$

$= 0, \qquad \text{if } x \le x_k$

Consider a third order linear boundary value problem of the form

$$y'''(x)+p(x)\,y''(x)+q(x)\,y'(x)+r(x)\,y(x)=m(x)\quad a\le x\le b. \tag{33}$$

with the boundary conditions

$$\begin{aligned}
&\alpha_0\,y_0+\beta_0\,y_n'+\gamma_0\,y_n''=\delta_0\\
&\alpha_1\,y_0'+\beta_1\,y_n+\gamma_1\,y_n''=\delta_1\\
&\alpha_2\,y_0''+\beta_2\,y_n+\gamma_2\,y_n'=\delta_2
\end{aligned} \tag{34}$$

Where $y(x), p(x), q(x), r(x), m(x)$ are continuous functions defined in the interval $x\in[a,b]$; $\delta_0, \delta_1, \delta_2$ are finite real constants.

Let equation (32) be an approximate solution of equation (33), where a_0, b_0, c_0, d_0, e_0, e_1, …, e_{n-1} are real coefficients to be determined.

Let $x_0, x_1, \ldots x_n$ be n+1 grid points in the interval [a,b], so that

$$x_i = a+ih,\ \text{i=0,1,…n};\ x_0=a,\ x_n=b\ \&\ \text{h=(b-a)/n}. \tag{35}$$

It is required that the approximate solution (32) satisfies the differential equation at the point $x=x_i$. Putting (32) in (33), we obtain the collocation equations as follows:

$$\begin{aligned}
&\sum_{k=0}^{n-1} e_k\left\{(x_i-x_k)_+ + \frac{1}{2}p(x_i)(x_i-x_k)_+^2+\frac{1}{6}q(x_i)(x_i-x_k)_+^3+\frac{1}{24}r(x_i)(x_i-x_k)_+^4\right\}\\
&+d_0\left\{1+p(x_i)(x_i-x_0)+\frac{1}{2}q(x_i)(x_i-x_0)^2+\frac{1}{6}r(x_i)(x_i-x_0)^3\right\}\\
&+c_0\left\{p(x_i)+q(x_i)(x_i-x_0)+\frac{1}{2}r(x_i)(x_i-x_0)^2\right\}\\
&+b_0\left\{q(x_i)+r(x_i)(x_i-x_0)\right\}+a_0\left\{r(x_i)\right\}=m(x_i)
\end{aligned} \tag{36}$$

Where i = 0, 1, 2,…. n.

From boundary conditions,

$$\sum_{k=0}^{n-1} e_k \left(\frac{\beta_0}{6}(b-x_k)_+^3 + \frac{\gamma_0}{2}(b-x_k)_+^2 \right) + d_0 \left(\frac{\beta_0}{2}(b-a)^2 + \gamma_0(b-a) \right) \quad (37)$$
$$+c_0\left(\beta_0(b-a)+\gamma_0\right)+b_0\left(\beta_0\right)+a_0(\alpha_0)=\delta_0$$

$$\sum_{k=0}^{n-1} e_k \left(\frac{\beta_1}{24}(b-x_k)_+^4 + \frac{\gamma_1}{2}(b-x_k)_+^2 \right) + d_0 \left(\frac{\beta_1}{6}(b-a)^3 + \gamma_1(b-a) \right) \quad (38)$$
$$+c_0\left(\frac{\beta_1}{2}(b-a)^2+\gamma_1\right)+b_0\left(\beta_1(b-a)+\alpha_1\right)+a_0(\beta_1)=\delta_1$$

and

$$\sum_{k=0}^{n-1} e_k \left(\frac{\beta_2}{24}(b-x_k)_+^4 + \frac{\gamma_2}{6}(b-x_k)_+^3 \right) + d_0 \left(\frac{\beta_2}{6}(b-a)^3 + \frac{\gamma_2}{2}(b-a)^2 \right) \quad (39)$$
$$+c_0\left(\frac{\beta_2}{2}(b-a)^2+\gamma_2(b-a)+\alpha_2\right)+b_0\left(\beta_2(b-a)+\gamma_2\right)+a_0(\beta_2)=\delta_2$$

Using the power function $(x - x_k)_+$ in the above equations a system of n+4 linear equations in n+4 unknowns a_0, b_0, c_0, d_0, e_0, e_1, …, e_{n-1} is thus obtained. This system can be written in matrix-vector form a follows

$$AX=B \quad (40)$$

where $X= [e_{n-1}, e_{n-2}, \dots, e_2, e_1, e_0, d_0, c_0, b_0, a_0]^T$

$$B=\left[\delta_2,\delta_1,\delta_0,m(x_n),m(x_{n-1}),\dots,m(x_1),m(x_0)\right]^T$$

The coefficient matrix A is an upper triangular Hessenberg matrix with a single lower sub diagonal, principal and upper diagonal having non-zero elements. Because of this nature of matrix A, the determination of the required quantities becomes simple and consumes less time. The values of these constants ultimately yield the Quartic spline s(x) in equation (32).

In case of nonlinear boundary value problem, the equations can be converted into linear form by Quasilinearization method [Bellman et al (9)] and hence this method can be used as iterative method. The procedure to obtain a spline approximation of y_i(i=0, 1, 2… j; where j denotes the number of iteration) by an iterative method starts with fitting a curve satisfying the end conditions and this curve is designated as y_i. We obtain the successive iterations y_i's with the help of an algorithm described as above till desired accuracy.

SOLUTIONS USING QUARTIC SPLINE COLLOCATION METHOD

Here the quartic spline method is employed to study the flow of a fluid which is governed by equation (28). Our aim is to see the reliability of spline solution and for that, we obtain the spline solutions of the problem over a domain [0,1]. Thus the boundary condition at the second end becomes w'(1)=1. Using this condition in the Maclaurin's series we find the value of α as α=1.435687792.

An application of Quasilinearization method to the nonlinear boundary value problem (28) is made to convert it in a linear one. Then the equation (28) for w_{i+1} is described as

$$w_{i+1}''' + w_i w_{i+1}'' - 2w_i' w_{i+1}' + w_i'' w_{i+1} = w_i w_i'' - \left(w_i'\right)^2 - 1 \qquad (41)$$

with the boundary conditions

$$w_{i+1}(0) = 0,$$

$$w_{i+1}'(0) = 0$$

and $$w_{i+1}(1) = 1 \qquad (42)$$

In order to obtain a quartic spline approximation, we begin with a parabola $w(\xi) = a\xi^2 + b\xi + c$ satisfying the conditions (42), as an initial guess.

The parabola is found to be

$$w^{(0)}(\xi) = 0.5\,\xi^2$$

Taking *N* equal subintervals of the domain [0, 1] a quartic spline for this problem is given by

$$s(\xi) = a_0 + b_0\xi + \frac{1}{2}c_0\xi^2 + \frac{1}{6}d_0\xi^3 + \frac{1}{24}\sum_{k=0}^{N-1} e_k\left(\xi - \xi_k\right)_+^4 \qquad (43)$$

Assume the above be an approximate solution to $w(\xi)$ of equation (41). Substituting $s(\xi)$ with its subsequent expressions in equations (41) and (42), we get the system of equations as follows:

$$\sum_{k=0}^{N-1} e_k \left\{ (\xi_i - \xi_k)_+ + \frac{w_i}{2}(\xi_i - \xi_k)_+^2 - \frac{w_i^{'}}{3}(\xi_i - \xi_k)_+^3 + \frac{w_i^{''}}{24}(\xi_i - \xi_k)_+^4 \right\}$$

$$+d_0 \left\{ 1 + w_i(\xi_i - \xi_0) - w_i^{'}(\xi_i - \xi_0)^2 + \frac{w_i^{''}}{6}(\xi_i - \xi_0)^3 \right\} \quad (44)$$

$$+c_0 \left\{ w_i - 2w_i^{'}(\xi_i - \xi_0) + \frac{w_i^{''}}{2}(\xi_i - \xi_0)^2 \right\} = w_i w_i^{''} - w_i^{'2} - 1,$$

$$i = 0, 1, 2, \ldots N$$

because the first two boundary conditions of (42) instantly gives $a_0 = 0$ & $b_0 = 0$ and $at\ \eta = 1,$ we obtain

$$c_0\xi_N + \frac{1}{2}d_0\xi_N^2 + \frac{1}{6}\sum_{k=0}^{N-1} e_k\ (\xi_N - \xi_k)_+^3 = 1 \quad (45)$$

where N is the number of subintervals of [0,1] and $c_0, d_0, e_0, e_1, \ldots e_{N-1}$ are to be determined.

The solutions thus obtained for *w* & *w*' are shown in graphical form, which shows the velocity of fluid flow w and w'. The results exhibited are obtained after three iterations only. Comparisons of the results with the solutions obtained by Maclaurin's series are shown in the following graphs.

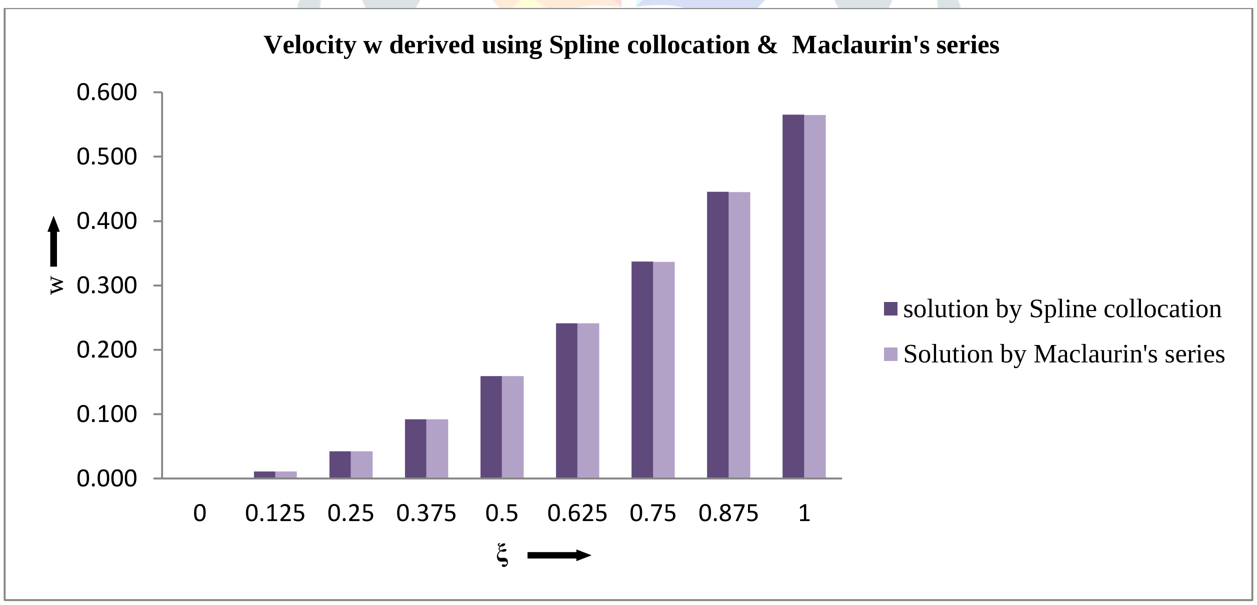

Fig. 5: *comparison of Velocity w using Spline Collocation method & Maclaurin's series*

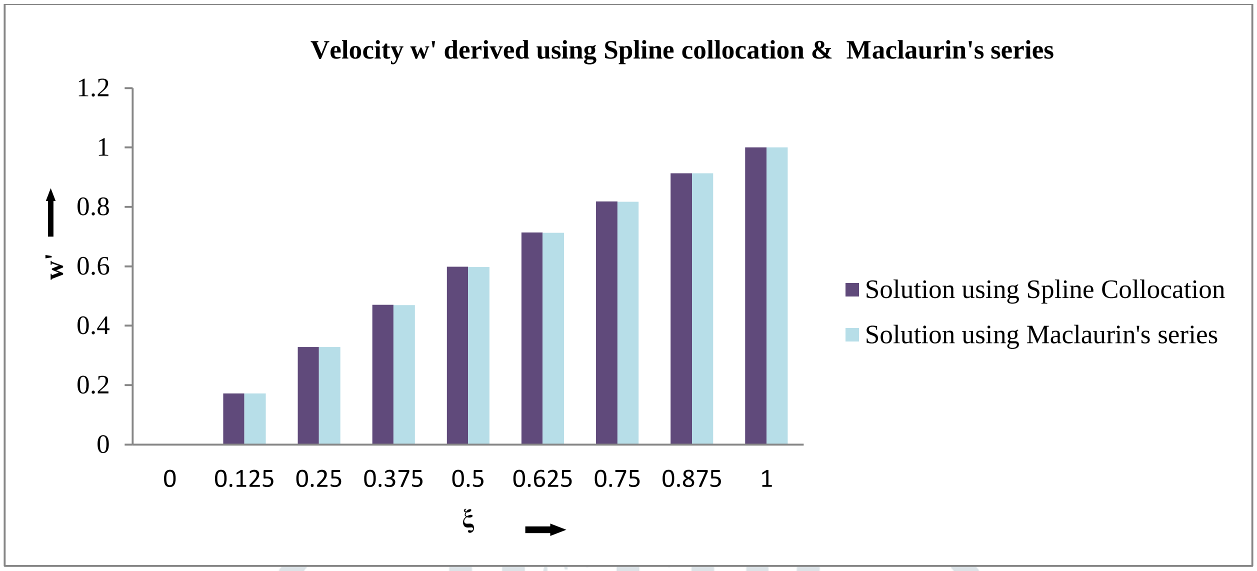

***Fig. 6:** comparison of Velocity w' using Spline Collocation method & Maclaurin's series*

DISCUSSION OF RESULTS AND CONCLUSIONS

The solutions of the problem are obtained in the form of velocity profile, which are shown by graphs. The represented accurate results are obtained using Spline collocation method after three iterations only. The close proximity of the spline solutions indicates the faster convergence. In order to obtain better results the reduction of length of subintervals of the domain is recommended. Comparison of the results with the results obtained by Maclaurin's series is quite encouraging which justifies the applicability and proves the reliability of the method.

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