THE MERSENNE JOIN MATRICES ON A-SETS

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ABSTRACT

The join matrix $[S]_f$ on S with respect to a function $f : P \to \mathbf{C}$ is defined as $([S]_f)_{ij} = f(x_i \vee x_j)$.If $f(x_i \vee x_j) = 2^{x_i \vee x_j}-1$, then the n × n join matrix obtained is called the Mersenne join matrix on S.
A recursive structure theorem for Mersenne join matrices on *A*-sets is verified and a recursive formula for $\det[S]_f$ and for $[S]_f^{-1}$ on *A*-sets is also verified. The recursive formulaealso yield explicit formulae, e.g. the known determinant and inverse formulae on chainsand *a*-sets.

Key words : Join Matrices, Mersenne Join Matrices, a- Set, A-Set

INTRODUCTION

Let $(P,\leq) = (P,\vee)$ be a join semi lattice, let $S = (x_1, x_2, \ldots\ldots\ldots\ldots x_n)$ be a subset of P and let $f : P \to \mathbf{C}$ be a function. The join matrix $[S]_f$ on S with respect to a function f is defined as $([S]_f)_{ij} = f(x_i \vee x_j)$.We say that S is join-closed if $x \vee y \in S$ whenever $x, y \in S$. We say that S is upper-closedif $(x \in S, x \leq y) \Rightarrow y \in S$ holds for every $y \in P$. It is clear that an upper-closed set is always join-closed but the converse does not hold.

In [10] we introduced join matrices and presented formulae for $\det[S]_f$, new upper and lower bounds for $\det[S]_f$ and a new formula for $[S]_f^{-1}$ on join-closed sets S (i.e.,$x_i,x_j \in S \Rightarrow x_i \vee x_j \in S$).By assuming the semi-multiplicativityof$f$, formulae for $\det[S]_f$ and $[S_f]^{-1}$ on join-closed sets are also presented in [10].

We say that S is an ***A*-set** if the set $A = \{x_i \vee x_j / x_i \neq x_j\}$ is a chain. For example, chains and *a*-sets (with $A = \{a\}$ are known trivial *A*sets.Since the method, presented in [12], adapted to *A*-sets might not be sufficiently effective, we give a new structure theorem for $[S]_f$ where S is an *A*-set. One of its featuresis that it supports recursive function calls.

By the structure theorem we obtain a recursive formula for $\det[S]_f$ and for$[S]_f^{-1}$ on *A*-sets. By dissolving the recursion on certain sets we also obtain e.g. the known explicitdeterminant and inverse formulae on chains and *a*-sets. We also briefly list the dualforms of our results, i.e. the structure theorem, determinant formulae and the inverse formulaefor join matrices on join-semilattices.

Note that $(\mathbf{Z}+,|) = (\mathbf{Z}+,\gcd,\mathrm{lcm})$ is a locally finite lattice, where |is the usual divisibility relation and gcd and lcm stand for the greatest common divisor and the least common multiple of integers.Thus join matrices are generalizations of LCM matrices $([S]_f)_{ij} = f(\mathrm{lcm}(x_i,x_j))$ and therefore theresults in this paper also hold for LCM matrices. For general accounts of LCM matrices, see [6] and [10, Section 6]. Join matrices are also generalizationsof LCUM matrices, the unitary analogies of LCM matrices, see [8]. Thus the results also hold for LCUM matrices (provided that we defineLCUM matrices as done in [8]).

DEFINITIONS

Let $(P,\leq) = (P,\vee)$ be a join-semilattice and let S be a nonempty subset of P. We say that S is join-closed if $x\vee y\in S$ whenever $x, y\in S$.

The method used requires that we arrange the elements of S analogously to the elements of chain A. Thus we give a more applicable definition for a-sets and A-sets than were seen in Introduction.

Definition 2.1 *The binary operation $\sqcup$ is defined by*

$$S_1\sqcup S_2 = \{ x \vee y \,/\, x \in S_1, y \in S_2, x \neq y\} \quad (2.1)$$

*where S_1 and S_2 are nonempty subsets of P. Let S be a subset of P and let $a \in P$. If $S \sqcup S = \{a\}$, then the set S is said to be an **a-set**.*

Definition 2.2 *Let* $S = \{x_1, x_2,\ldots\ldots\ldots\ldots x_n\}$ *be a subset of and let* $A= \{a_2,a_3,\ldots\ldots\ldots\ldots a_n\}$ *be a multichain (i.e. a chain where duplicates are allowed) . The set S is said to be an **A-set** if $\{x_1,x_2\ldots\ldots, x_{k-1}\}\sqcup\{x_k\}= \{a_k\}$ for all $k = 2,3,\ldots\ldots, n$.*

Every chain $S = \{x_1, x_2,\ldots\ldots\ldots\ldots x_n\}$ is an A-set with $A = S \setminus\{x_n\}$ and every a-set is always an A-set with $A = \{a\}$.

Definition 2.3 *Let f be a complex-valued function on P. Then the $n \times n$ matrix $[S]_f$, where $([S]_f)_{ij}= f(x_i\vee x_j)$, is called the join matrix on S with respect to f. Also the $n \times n$ matrix $[S]_f$, where $([S]_f)_{ij}= f(x_i\vee x_j) =2^{x_i\vee x_j}-1$, is called the Mersenne join matrix.*

In what follows, let $S = \{x_1, x_2,\ldots\ldots\ldots\ldots x_n\}$ always be a finite subset of P . Let also $A = \{a_2, a_3,\ldots\ldots\ldots\ldots a_n\}$. Note that S has always n distinct elements, but it is possible that the set A is a multiset. Let f be a complex-valued function on P.

3 MERSENNE JOIN MATRICES ON *A*-SETS

3.1 Structure Theorem

Theorem 3.1 (Structure Theorem) *Let* $S = \{x_1, x_2,\ldots\ldots\ldots\ldots x_n\}$ *be a finite set such that* $\{x_1,x_2\ldots\ldots, x_{k-1}\} \sqcup\{x_k\} = \{a_k\}$ *for all* $k = 2,3,\ldots\ldots, n$, *where* $A =\{a_2, a_3,\ldots\ldots\ldots\ldots a_n\}$ *is a multichain. Let* $f_1, f_2,\ldots\ldots, f_n$ *denote the functions on P defined by* $f_1 = f$ *and*

$$f_{k+1}(x) = f_k(x) - \frac{f_k(a_{n-k+1})^2}{f_k(x_{n-k+1})} \quad (3.1)$$

for $k = 1, 2,\ldots\ldots\ldots, n - 1$.

Then

$$[S]_f= M^TDM, \quad (3.2)$$

where $D = \text{diag}\,(f_n(x_1), f_{n-1}(x_2),\ldots\ldots\ldots, f_1(x_n))$ *and M is the $n \times n$ lower triangular matrix with 1's on its main diagonal, and further*

$$(M)_{ij}= \frac{f_{n-i+1}(a_i)}{f_{n-i+1}(x_i)} \quad (3.3)$$

for all $i> j$.

.

Proof : Let $i> j$. Then

$$(M^TDM)_{ij}=\sum_{k=1}^{n} (M)_{ki}(D)_{kk}(M)_{kj}= f_i(a_i) +\sum_{k=1}^{i-1}\frac{f_k(a_k)^2}{f_k(x_k)} \quad (3.4)$$

$$= f_i(a_i) +\sum_{k=1}^{i-1} (f_k(a_i) - f_{k+1}(a_i)) = f_1(a_i) = f(x_i\vee x_j).$$

The case $i=j$ is similar, we only replace every a_i with x_i in (3.4). Since M^TDM issymmetric, we do not need to treat the case $i<j$.

3.2 Determinant of Join matrix on *A*-sets

By Structure Theorem we obtain a new recursive formula for det$[S]_f$ on*A*-sets.

Theorem 3.2 *Let S*=$\{x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$*be an A-set such that* $\{x_1, x_2, \ldots, x_{k-1}\} \sqcup \{x_k\} = \{a_k\}$*where* $A=\{a_2, a_3, \ldots\ldots\ldots\ldots a_n\}$ *is amultichain. Let* $f_1, f_2, \ldots\ldots, f_n$*be the functions defined in* (3.1). *Then*

det $[S]_f = f_n(x_1) f_{n-1}(x_2) \ldots\ldots\ldots f_1(x_n)$,(3.5)

Corollary 3.1 *If* $S = \{x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$*is a chain, then*

Det $[S]_f = f(x_n)\prod_{k=2}^{n}(f(x_{k-1}) - f(x_k)$(3.6)

Proof :By Theorem 3.2 we have

Det$[S]_f = f_1(x_1)f_2(x_2)\ldots\ldots\ldots f_n(x_n)$, where $f_1 = f$ and

$f_{k+1}(x) = f_k(x) - f_k(x_k) = f(x) - f(x_k)$ for all $k = 1,2,\ldots\ldots\ldots\ldots\ldots, n-1$. This completes the proof.

By Theorem 3.2 we also obtain a known explicit formula for det$[S]_f$on*a*-sets. This formula has been presented (with different notation) in [4, Corollary of Theorem 3] and [12,Corollaries 5.1 and 5.2], and also in [2, Theorem 3] in number-theoretic setting. Note that the case $f(a) = 0$ is trivial, since then $[S]_f = \mathrm{diag}(f(x_1), f(x_2), \ldots\ldots., f(x_n))$ anddet$[S]_f = f(x_1)f(x_2)\ldots\ldots f(x_n)$.

Corollary 3.2

Let $S=\{x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$ be a set such that $x_i \vee x_j = a$ whenever$x_i \neq x_j$ *and let* $f(a) \neq 0$. *If* $a \in S$ *(i.e.* $a = x_n$*), then*

det$[S]_f = (f(x_1) - f(a))\ldots\ldots(f(x_{n-1}) - f(a))f(a)$.(3.7)

If $a \notin S$*, then*

$$\mathrm{Det}[S]_f = \sum_{k=1}^{n} \frac{f(a)\big(f(x_1) - f(a)\big)\ldots\ldots\big(f(x_n) - f(a)\big)}{f(x_k) - f(a)} + (f(x_1) - f(a))\ldots.(f(x_n) - f(a)). \qquad (3.8)$$

Example 3.1 Let $(P,\leq) = (\mathbf{Z}+,|)$ and $S=\{1,3,6\}$.

Then $S=\begin{bmatrix} 2^1-1 & 2^3-1 & 2^6-1 \\ 2^3-1 & 2^3-1 & 2^6-1 \\ 2^6-1 & 2^6-1 & 2^6-1 \end{bmatrix}$Since *S* is an *A*-set with the chain $A = \{3,6\}$ by (3.1) we have $f_1 = 2^x - 1$, $f_2(x) = f_1(x) - f_1(6)^2/f_1(6)$ and$f_3(x) = f_2(x) - f_2(3)^2/f_2(3)$.and. Let $f(x) = 2^x - 1$. Then $f_1(x) = 2^x - 1$,$f_2(x) = 2^x - 64$, $f_3(x) = 2^x - 8$

andby Theorem 3.1 $[S]_f = M^TDM$, where D=diag(-6,-56,63)and $M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

and by Theorem 3.2 we have det$[S]_f = f_3(1)f_2(3)f_1(6) = (-6)(-56)(63) = 21168$.

3.3 Inverse of Mersenne join matrix on *A*-sets

By Structure Theorem we obtain a new recursive formula for $[S_f]^{-1}$on *A*-sets.

Theorem 3.3 *Let* $S = \{x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$ *be an A-set, where* $A = \{a_2, a_3, \ldots\ldots\ldots a_n\}$ *is a* *multichain. Let* $f_1, f_2, \ldots, f_n$ *be the functions defined in* (3.1), *where* $f_{n-i+1}(x_i) \neq 0$ *for* $i=1,2,\ldots,n$. *Then* $[S]_f$ *is invertible and* $[S_f]^{-1} = N\triangle N^T$ (3.9)

where $\triangle = \mathrm{diag}(1/f_1(x_1), 1/f_2(x_2), \ldots, 1/f_n(x_n))$ *and N is the n×n lower triangular matrix with1's on its main diagonal, and further*

$$(N)_{ij} = -\frac{f_{n-i+1}(a_i)}{f_{n-i+1}(x_i)} \prod_{k=j-1}^{i+1} \left(1 - \frac{f_{n-k+1}(a_k)}{f_{n-k+1}(x_k)}\right) \quad (3.10)$$

for all $i > j$.

*Proof*By Structure Theorem $[S]_f = M^T DM$, where M is the matrix defined in(3.3) and $D = \mathrm{diag}(f_n(x_1), f_{n-1}(x_2), \ldots, f_1(x_n))$. Therefore $[S_f]^{-1} = N\triangle N^T$, where $D^{-1} = \mathrm{diag}(1/f_n(x_1), 1/f_{n-1}(x_2), \ldots, 1/f_1(x_n))$ and $M^{-1} = N$ is the $n \times n$ *lowe*r triangular matrix in (3.10).

Example 3.1.1

S is considered the same as in Example 3.1 then by $[S_f]^{-1} = N\triangle N^T$,

$\triangle = \mathrm{diag}(1/-6, -1/56, 1/63)$), $N = M^{-1}$, $N = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$,

$$[S_f]^{-1} = \begin{bmatrix} -1/6 & 1/6 & 0 \\ 1/6 & -62/336 & 1/56 \\ 0 & 1/56 & -1/504 \end{bmatrix}$$

Corollary 3.3 *Let* $S = \{x_1, x_2, \ldots\ldots x_n\}$ *be an a-set, where* $f(a) \neq 0$ *and* $f(x_k) \neq f(a)$ *for all* $k = 2, \ldots, n-1$. *If* $a \in S$ *(i.e.* $a = x_1$*), then* $[S]_f$ *is invertible and*

$$([S_f]^{-1})_{ij} = \begin{cases} \frac{1}{f(x_i)-f(a)} \; if\ i=j<1, \\ \frac{1}{f(a)} + \sum_{k=1}^{n-1} \frac{1}{f(x_k)-f(a)} = n, \\ \frac{1}{f(a)-f(x_k)} \quad if\ k=i<j=n\ or\ k=j<i=n \\ 0 \qquad\qquad otherwise \end{cases} \quad (3.11)$$

If $a \notin S$ *and further* $\frac{1}{f(a)} \neq \sum_{k=1}^{n-1} \frac{1}{f(x_k)-f(a)}$, *then* $[S]_f$ *is invertible and*

$$([S_f]^{-1})_{ij} = \begin{cases} \frac{1}{f(x_i)-f(a)} - \frac{1}{[f(x_i)-f(a)]^2}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & if\ i=j, \\ \frac{-1}{[f(x_i)-f(a)][f(x_j)-f(a)]}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & if\ i \neq j. \end{cases} \quad (3.12)$$

CONCLUSION:

In this paper we prove by examples that the MersenneJoin matrices on A sets satisfies structure theorem and calculate the determinant and inverse of theMersenneJoin matrix through results based on A sets.

References:

[1] M. Aigner, *Combinatorial Theory*. Springer-Verlag, 1979.

[2] S. Beslin and S. Ligh, GCD-closed sets and the determinants of GCD matrices, *Fibonacci Quart.*, 30: 157-160 (1992).

[3] G. Birkhof, *Lattice Theory*. American Mathematical Society Colloquium Publications, 25, Rhode Island, (1984).

[4] P. Haukkanen, On meet matrices on posets, *Linear Algebra Appl.* 249: 111-123 (1996).

[5] P. Haukkanen and J. SillanpÄaÄa, Some analogues of Smith's determinant, *Linear and Multilinear Algebra* 41: 233-244 (1996).

[6] P. Haukkanen, J.Wang and J. SillanpÄaÄa, On Smith's determinant, *Linear Algebra Appl.* 258: 251-269 (1997).

[7] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[8] I. Korkee, A note on meet and join matrices and their special cases GCUD and LCUM Matrices,*Int. J. Pure Appl. Math.1-11*

[9] I. Korkee and P. Haukkanen, Bounds for determinants of meet matrices associated with incidence functions, *Linear Algebra Appl.* 329(1-3): 77-88 (2001).

[10] I. Korkee and P. Haukkanen, On meet and join matrices associated with incidence functions. *Linear Algebra Appl.*, 372: 127-153 (2003).

[11] I. Korkee, On meet and join matrices on *A*-sets and related sets, Notes on Number theory and Discrete Mathematics, 10(3):57-67,(2004)

[12] I. Korkee and P. Haukkanen, On meet matrices with respect to reduced, extended and exchanged sets, *JP J. Algebra Number Theory Appl.*, 4(3),559-575 (2004)

[13]B. V. RajaramaBhat, On greatest common divisor matrices and their applications, *Linear Algebra Appl.* 158: 77-97 (1991).

[14]B. Wang, Explicit Expressions of Smith's Determinant on a Poset, *Acta Math. Sin. (Engl. Ser.)*, 17(1): 161-168 (2001).

[15] F. Zhang, Matrix theory. Basic results and techniques, Universitext, Springer-Verlag, New York, 1999.