Complementary Tree Domination in Subdivision graphs

Dr.P.Vidhya
EMG Yadava Women's College,
Madurai- 625 014,India

S.Jayalakshmi
S.D.N.B Vaishnav College for Women (Autonomous)
Chennai-600 044,India

Abstract

A set D of a graph G = (V,E) is a dominating set if every vertex in V-D is adjacent to some vertex in D .The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. A dominating set D is called a complementary tree dominating set if the induced sub graph $\langle V-D\rangle$ is a tree. The minimum cardinality of a complementary tree dominating (ctd) set is called the complementary tree domination number of G and it is denoted by $\gamma_{ctd}(G)$. A subdivision of an edge $e=uv$ of a graph G is the replacement of the edge e by a path (u,v,w). The graph obtained from G by sub dividing every edge e of G exactly once, is called the subdivision graph of G and is denoted by $S(G)$. In this paper, exact values of some standard graphs and bounds of complementary tree domination number in $S(G)$ are found.

Mathematics Subject Classification: 05C69

Keywords: domination number, subdivision graphs

1 Introduction

Graphs discussed in this paper are undirected and simple graphs .For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. For $v\in V(G)$, the neighbourhood $N(v)$ of v is the set of all vertices adjacent to v in G. $N[v]=N(v)\cup\{v\}$ is called the closed neighbourhood of v. A vertex $v\in V(G)$ is called a support if it is adjacent to a pendant vertex (ie) a vertex of degree one. The graph considered here are finite, undirected, without loops or multiple edges are connected with p vertices and q edges.

The concept of domination in graphs was introduced by Ore[4]. A set $D\subseteq V(G)$ is said to be a dominating set of G ,if every vertex in $V(G)-D$ is adjacent to some vertex in D. D is said to be a minimal dominating set .

Definition 1.1. A set D$\subseteq$ $V(G)$ is said to be a complementary tree dominating set (ctd- set) if the induced sub graph $\langle V(G)-D\rangle$ is a tree. The minimum cardinality of a ctd -set is called the complementary tree domination number of G and it is denoted by $\gamma_{ctd}(G)$.

Definition 1.2. A subdivision of an edge $e = uv$ of a graph G is the replacement of the edge e by a path (u, v, w) . The graph obtained from G by sub dividing every edge e of G exactly once, is called the subdivision graph of G and is denoted by $S(G)$.

2 . Characterisation of Complementary Tree Dominating Sets in Subdivision Graph S(G)

We start with some basic results

Observation 2.1.

1.For any connected graph G, $\gamma_{ctd}(G) \le \gamma_{ctd}[S(G)]$

2.For any spanning sub graph $S(H)$ of $S(G)$, $\gamma_{ctd}[S(G)] \le \gamma_{ctd}[S(H)]$

Preposition 2.2. Atmost $(p-1)$ vertices of $V(G)$ is a member of every ctd set.

Proof. Let $e = (u, v) \in V(G)$ and let v_1 be a vertex subdivide e, then $u, v, v_1 \in S(G)$.Let D be a ctd set of $S(G)$. If $v_1 \in D$ then $\langle V - D \rangle$ is disconnected which is a contradiction. Therefore either $u \text{ or } v \in D$

Theorem 2.3. A complementary tree dominating set $D \subseteq V(S(G))$ of a connected graph $G = (V, E)$ is minimal if and only if for each $v \in D$ and in a vertex of $V(G)$,one of the following condition holds.

i. v is not a isolated vertex of D.

ii. There exists a vertex u in $V(S(G)) - D$ such that $N_2(u) \cap D = \{v\}$.

iii. $N(v) \cap (V(S(G)) - D) = \phi$.

iv. $D - \{v\}$ contains isolate vertex.

v. The sub graph $\langle V(S(G)) - D \rangle \cup \{v\}$ of $S(G)$ is disconnected.

3. Bounds and some exact values of $\gamma_{ctd} S(G)$

Observations 3.1.

i. $\gamma_{ctd}(S(C_n)) = 2p - 2, \quad p \ge 3$

ii. $\gamma_{ctd}(S(P_n)) = 2p - 3, \quad p \ge 2$

iii. $\gamma_{ctd.}(S(w_p)) = 2p - 1, \quad p \ge 4$

$= q + 1$

iv. $\gamma_{ctd}(S(K_{1,p-1})) = p$

$= q + 1$

v. $\gamma_{ctd}(S(K_p)) = \dfrac{p^2 - p + 2}{2}$

$= q + 1$

vi. $\gamma_{ctd}(S(K_{m,n})) = m + mn, \quad m \le n$

vii. $\gamma_{ctd}\left(S\left(c_p \circ K_1\right)\right) = 4p-3$

$= 2q+3-3$

Theorem 3.2. For any connected graph G $p \geq 2$, then $\gamma_{ctd}\left(S(G)\right) \geq 2$.

Proof. Every complementary tree dominating set of $S(G)$ contains at least one vertex of $V(G)$ and $V(S(G)) - V(G)$.

Therefore,

$$\gamma_{ctd}\left(S(G)\right) \geq 2.$$

Theorem 3.3. If $\gamma_{ctd}(G) = 2$ *if and only if* $G \cong K_2$.

Proof. Assume $G \cong K_2$. Let $u, v \in E(G)$ and w be the vertex in $S(G)$ such that w is adjacent to u and v. Then $\{u, w\}$ is a ctd set of $S(G)$ and hence $\gamma_{ctd}\left(S(G)\right) = 2$.

Conversely, if $\gamma_{ctd.}\left(S(G)\right) = 2$, then there exist a ctd set of D of $S(G)$ with $|D| = 2$ and let $D = \{u, v\}$ such that $\langle V(S(G)) - D \rangle$ is a tree.

Case (i):

$$|V(S(G)) - D| = 1$$

Let w be the vertex of $V(S(G)) - D$ then w is either adjacent to any one of the vertex of D or adjacent to both. Therefore $S(G) \cong P_3$ (ie) $G \cong K_2$.

Case(ii):

$$|V(S(G)) - D| > 1$$

Let v_1 *and* v_2 be the vertices of $\langle V(S(G)) - D \rangle$ .Then v_1 *and* v_2 are adjacent to any one of the vertices of D. Without loss of generality v_1 is adjacent to u and v_2 is adjacent to v then $S(G) \cong p_4$ which a contradiction is. For all values of $n \geq 2$, $S(G) \cong P_{2n+1}$. Therefore $G \cong K_2$.

Theorem 3.4. For any connected graph G of order $p \geq 3, \gamma_{ctd}\left(S(G)\right) \leq 2p-2$,

Also $\gamma_{ctd}\left(S(G)\right) = 2p-2$ if and only if $G \cong C_p$.

Proof. Let $\{uv, vw\} \in E(G)$ and let x, y are the vertices in $S(G)$ such that x is subdivides uv and y is subdivides vw. Then $V(S(G)) - \{v, x\}$ is a ctd set of $S(G)$ and hence

$$\gamma_{ctd}\left(S(G)\right) = 2p-1-2$$
$$= 2p-3$$
$$< 2p-2$$

Suppose G contains a cycle C_p with edge set $E(G)=\{uv, vw, wx, ...yu\}$ and let $\{x_1, x_2, ...x_p\}$ be the vertices in $S(G)$ such that $x_1, x_2, ...x_p$ subdivides $uv, vw, wx, ...yu$ respectively. Then $V(S(G))-\{u,x\}$ is a ctd set of $S(G)$ and hence $\gamma_{ctd}(S(G))=2p-2.$

Conversly, assume $G \cong C_p \quad, p \geq 3.$

We know $\gamma_{ctd}(C_p)=p-2$

$$\gamma_{ctd}(S(C_p))=\gamma_{ctd}(C_{2p})$$

$$=2p-2.$$

Theorem 3.5. For the complete graph K_p then $\gamma_{ctd}(S(K_p))=\frac{p^2-p+2}{2}.$

Proof. The result is true if $p=2.$

Suppose $p \geq 3,$ Let D be a minimum ctd set of $S(K_p)$. Let $V(K_p)=\{v_1, v_2, ...v_p\}$ and $W=V(S(K_p))-V(K_p)$

$=\{w_1, w_2 ....w_r\}$, where $r=\binom{p}{2}$ without loss of generality, we may assume that

$D=\{v_1, w_1, v_2, w_2 ....v_{p-1}, w_{p-1}, ...w_{p+1}, w_{p+2} ....w_{r-p+3}\}$

therefore

$$|D| = p-1+\frac{p(p-1)}{2}-(p-2)$$

$$=\frac{p^2-p+2}{2}$$

or

$$=q+1.$$

Theorem 3.6. For the Complete bipartite graph $K_{m,n}$ $, m \leq n$ then $\gamma_{ctd}(S(K_{m,n}))=m+mn.$

Proof. Let $V_1=\{u_1, u_2, ...u_m\}$ and $V_2=\{v_1, v_2, ...v_n\}$ be a bipartition of $K_{m,n}$. Let w_{ij} $(1 \leq i \leq m, 1 \leq j \leq n)$ be the vertex of $S(K_{m,n})$ which is adjacent to u_i and v_j .Without loss of generality we may assume

$D=\left(\bigcup_{i=1}^{m} u_i\right) \cup \left(\bigcup_{i=1}^{m} \bigcup_{j=1}^{n} w_{ij}\right) \cup \left(\bigcup_{j=1}^{n} v_j\right) - \bigcup_{j=1}^{n-1} v_j$ is a minimum ctd set of $S(K_{m,n})$.

$$\therefore \quad |D| = m-1+mn+n-(n-1)$$

$$=m-1+mn+n-n+1$$

$$=m+mn.$$

Theorem 3.7. For the wheel $W_p = C_{p-1} + K_1$,then $\gamma_{ctd}(S(W_p)) = 2p-1$.

Proof. Let $u_0, u_1, \ldots\ldots. u_{p-1}$ be the vertices of the wheel W_p with $\deg(u_0) = p-1$, v_i be the vertices of $S(W_p)$ adjacent to u_0 and u_i .Let D be a minimum ctd set of $S(W_p)$ and w_i be the vertex subdividing the edge $u_i u_{i+1}$, $1 \le i \le n-2$ and v_i be the vertex adjacent to u_0 $\& u_i$.

Without loss of generality we may assume $D = \bigcup_{i=1}^{p-1} u_i \cup \bigcup_{i=1}^{p-1} w_i \cup$(one of the neighbourhood of u_0).

$= \mathrm{p} - 1 + \mathrm{p} - 1 + 1$

$= 2\mathrm{p} - 1$

$\gamma_{\mathrm{ctd}}(W_p) = 2p-1$.

Theorem 3.8. For the star graph $K_{1,p-1}$ then

$\gamma_{ctd}(S(K_{1,p-1})) = p$.

Proof. Let $\{v_0, v_1, v_2, \ldots\ldots v_{p-1}\}$ be the vertices of $K_{1,p-1}$,then u_i be the vertex subdivides $v_0 v_i$, $1 \le i \le p-1$. We know that the pendant vertices are members of ctd set. Let D be the minimum ctd set of $S(G.)$

$$D = \bigcup_{i=1}^{p-1} v_i \cup \{\text{one of the vertex of } u_i\}$$

$|D| = p - 1 + 1$

$= p$

$\therefore \gamma_{ctd} S(K_{1,p-1}) = p$

Theorem 3.9. If T is a tree T of order p which is not a star then

$m + s - 1 \le \gamma_{ctd}(S(T)) \le 2p - 3$ where S denotes the number of supports and m denote the number of pendant vertices of T.

Proof. Let $V_1 = \{u_1, u_2, \ldots u_m\}$ be the set of all pendant vertices of T. V_2 be the set of all supports of T and $V_3 = \{v_1, v_2, \ldots. v_m\}$, where v_i is the vertex subdividing the edge incident with u_i. Let D be the minimum ctd set of $S(T)$ should contain $|v_1 \cup v_3| - 1$

$\therefore |D| = |v_1 \cup v_3| - 1$

and hence $\gamma_{ctd} S(T) \ge |D| \ge m + s - 1$.

Now to prove the upper bound we know that $\gamma_{ctd}(T) \le p - 2$ Since

$V(S(T)) = 2p - 1$

therefore

$$\gamma_{ctd}(S(T)) \le 2p - 1 - 2$$
$$= 2p - 3$$

The lower bound equality holds in P_3 and upper bound equality holds in P_n.

Theorem 3.10. For any connected (p,q) graph G, $\gamma_{ctd}(S(G)) + \Delta(G) \le 2p = p + q.$

Proof. For any graph with p vertices, $\Delta(G) \le p - 1$

By observation,

$\gamma_{ctd}(G) + \Delta(G) = 2p$ if $G \cong C_p$

$$when \quad G \cong C_p \quad \gamma_{ctd} S(C_p) + \Delta(C_p) = 2p - 2 + 2$$
$$= 2p$$
$$= p + q$$
$$when \quad G \cong P_p \quad \gamma_{ctd} S(G) + \Delta(G) = 2p - 3 + 2$$
$$= 2p - 1$$
$$= p + p - 1$$
$$= p + q$$

$$When \quad G \cong K_p, \quad \gamma_{ctd} S(G) + \Delta(G) = q + 1 + p - 1$$
$$= p + q$$
$$When \quad G \cong W_p, \quad \gamma_{ctd} S(G) + \Delta(G) = q + 1 + p - 1$$
$$= p + q$$

Theorem 3.11. For a connected graph G $p \ge 2$,then $\gamma_{ctd}(S(G)) \ge \gamma_{ctd}(G) + \delta(G)$ and Also, $\gamma_{ctd}(S(G)) = \gamma_{ctd}(G) + \delta(G)$ iff $G \cong K_{1,p-1}$.

Proof. Assume $G \cong K_{1,p-1}$ then subdivides the edge set vv_i in G by w_i , $1 \le i \le p-1$. Let D be the minimal ctd of G and D^1 be the minimal ctd of $S(G)$. D contains all the vertices of G and contains atleast one members of $S(G)$.

Therefore

$$\left|D^1\right| \geq |D| + 1$$
$$= \gamma_{ctd}(G) + \delta(G)$$

therefore

$$\gamma_{ctd}(S(G)) \geq \gamma_{ctd}(G) + \delta(G)$$

Conversly, assume $\gamma_{ctd}(S(G)) \geq \gamma_{ctd}(G) + \delta(G)$ .Suppose $\delta(G) = 1$,we know that

$$\gamma_{ctd}(G) \leq p - 1$$

therefore

$$\gamma_{ctd}(G) \leq p - 1 + 1$$
$$= p$$

therefore

$$G \cong K_{1,p-1}$$

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