AVERAGE TOTAL DOMINATION NUMBER ON GRAPHS

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Abstract

A network is modelled with the graphs in a situation which the centers are equal to the vertex of graphs and connection lines are equal to the edges of a graph. A graph G is denoted by G = (V (G);E(G)), where V (G) and E(G) are vertex and edge sets of G, respectively. Let v be a vertex in V (G) and p and q be the number of vertices and edges in G. In a graph G = (V (G);E(G)), a subset S ⊆ V (G) of vertices is a dominating set if every vertex in V (G)−S is adjacent to at least one vertex of S. The domination number γ(G) of a graph G is the minimum cardinality of a dominating set. The concept of total domination was introduced by Cockayne, Dawes and Hedetniemi. A dominating set S ⊆ V is a Total Dominating set if the induced subgraph <S> has no isolated vertices. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set. Henning introduced the concept of average domination number. The average domination number $\gamma_{ag}(G)$ is defined as $\frac{1}{|V(G)|}\sum_{v\in V}\gamma_V(G)$ where $\gamma_v(G)$ is the minimum cardinality of a dominating set that contains v. In this paper a new parameter namely average total domination number is defined and is studied for connected graphs. In this paper average total domination number is studied for complete binary tree. Some bounds on average total domination number in terms of total domination number are also established.

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1. Introduction

In a communication network, the vulnerability parameters measure the resistance of the network to disruption of operation after the failure of certain stations or communication links. Recent interest in the vulnerability and reliability of networks (communication, computer, and transportation) has given rise to a host of other measures, some of which are more global in nature. Communication network is modeled as simple, undirected, connected and unweighted graph G. Many graph theoretical parameters can be used to describe the stability and reliability of communication networks. If a graph is considered as a modeling network, the average domination number of a graph is one of the parameters for graph vulnerability. The average parameters have been found to be more useful in some circumstances than the corresponding measures based on worst–case situations. Henning [5] introduced the concept of average domination and independent domination numbers, studied trees for which these two parameters are equal. Derya Dogan [4] studied the average lower domination number for the middle graphs of some well known graphs. Further Ersin Aslan [1, 2] studied the average lower connectivity of several specific graphs. Let G = (V, E) be a simple graph of order p. For v ∈V(G), the neighborhood $N_G(v)$ (or simply N(v)) of v is the set of all vertices adjacent to v in G. The degree of a vertex is the size of its neighborhoods. The maximum degree of a graph G is denoted by Δ(G) and the minimum degree of a graph G is denoted by (G). The complete graph K_p, is a connected graph in which every pair of vertices are connected by an edge. In a connected graph

K_p all the vertices of degree p – 1. A pendant vertex in a graph G is a degree of vertex one and a vertex is called a support if it is adjacent to a pendant vertex. All the graphs considered here are simple, finite, undirected and connected graphs. In this paper average total domination number is studied for complete binary tree. Some bounds on average total domination number in terms of total domination number are also established.

2. Prior Results

The concept of domination in graphs was introduced by Ore [7].

Definition 2.1. A nonempty set $S \subseteq V(G)$ of a graph G is a **dominating set**, if every vertex in $V(G) – S$ is adjacent to some vertex in S. Every graph G has dominating sets. Every superset of a dominating set of G is a dominating set. But a subset of a dominating set need not be a dominating set.

A dominating set S of G is **minimal dominating set** if no proper subset of S is a dominating set. The **domination number** $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G. A dominating set of cardinality $\gamma(G)$ is called a γ–set of G.

Cockayne, Dawes and Hedetniemi [3] introduced the concept of total domination in graphs.

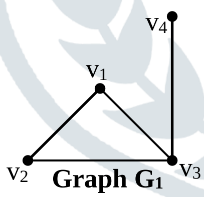
Definition 2.2. A dominating set S of a graph G without isolated vertices is called a **total dominating set** of G if <S> has no isolated vertices. The minimum cardinality of a total dominating set of G is called the **total domination number** of G and is denoted by $\gamma_t(G)$. A total dominating set of cardinality $\gamma_t(G)$ is called a γ_t–set of G.

Henning [5] introduced the concept of average domination.

Definition 2.3. The **lower domination number**, denoted by $\gamma_v(G)$ is the minimum cardinality of a dominating set of G that contains v. A dominating set of cardinality $\gamma_v(G)$ is called a γ_v–set of G.

Definition 2.4. The **average domination number** $\gamma_{ag}(G)$ is defined as $\frac{1}{|V(G)|}\sum_{v \in V} \gamma_v(G)$ where $\gamma_v(G)$ is the minimum cardinality of a dominating set that contains v.

Example 2.5. Consider the graph G_1 given in Figure 2.6.



Graph G_1
Figure 2.6.

The set $S = \{v_3\}$ forms a dominating set of G_1. Hence $\gamma(G_1) = 1$.
The set $S_1 = \{v_1, v_3\}$ forms a total dominating set of G_1. Hence $\gamma_t(G_1) = 2$.
The following table shows the minimum cardinality of a dominating set of G_1 that contains v

$v \in V(G_1)$	γ_v–set of G_1	$\gamma_v(G_1)$
v_1	$\{v_1, v_4\}$	2
v_2	$\{v_2, v_3\}$	2
v_3	$\{v_3\}$	1
v_4	$\{v_2, v_4\}$	2

$$\gamma_{ag}(G_1) = \frac{1}{|V(G_1)|}\sum_{v \in V} \gamma_v(G_1) = \frac{1}{4}(2 + 2 + 1 + 2) = \frac{7}{4}.$$

3. Main Result

Definition 3.1. The **lower total domination number**, denoted by $\gamma_{vt}(G)$ is the minimum cardinality of a total dominating set of G that contains v.

Definition 3.2. The **average total domination number** $\gamma_{agt}(G)$ is defined as $\frac{1}{|V(G)|}\sum_{v \in V} \gamma_{vt}(G)$ where $\gamma_{vt}(G)$ is the minimum cardinality of a dominating set that contains v.

Note 3.3. The average domination numbers namely $\gamma_{ag}(G)$ and $\gamma_{agt}(G)$ need not be an integer.

Illustration 3.4. The following table shows the minimum cardinality of a total dominating set of G_1 that contains v

$v \in V(G_1)$	γ_{vt}–set of G_1	$\gamma_{vt}(G_1)$
v_1	$\{v_1, v_3\}$	2
v_2	$\{v_2, v_3\}$	2
v_3	$\{v_2, v_3\}$	2
v_4	$\{v_3, v_4\}$	2

$$\gamma_{agt}(G_1) = \frac{1}{|V(G_1)|}\sum_{v \in V} \gamma_{vt}(G_1) = \frac{1}{4}(2 + 2 + 2 + 2) = 2.$$

Theorem 3.5. Any super set of a total dominating set is a total dominating set of a graph G.

Proof. Let S be a total dominating set of the connected graph G. Then by the definition of a dominating set any vertex in V – S is adjacent to atleast one vertex in S. If any of vertex in V – S is included in S then let S_1 be the new vertex set obtained. Since S is a total dominating set and any vertex in V – S is adjacent to a vertex in S, it follows that S_1 is also a total dominating set. Hence the theorem.

In the following, an upper bound of the average total domination number is given.

Theorem 3.6. For any connected graph G, $\gamma_{agt}(G) < \gamma_t(G) + 1$.

Proof. Let G be a connected graph with $|V(G)| = p$. Let $S_1, S_2, \ldots, S_r$ be distinct $\gamma_t(G)$–sets of G with $|S_1| = |S_2| = \ldots = |S_r| = \gamma_t(G)$. Therefore $\gamma_{vt}(G) = |S_i|$, for every $v \in S_i$, i = 1, 2, …, r. If there exists a vertex u which does not belong to S_i, for all i = 1, 2, …, r, then $\gamma_{ut}(G) = |S_i| + 1$, since $S_i \cup \{u\}$, for any i = 1, 2, …, r is a total dominating set of G by Theorem 3.5. containing u. Let $|\cup S_i| = k$. Let A be the set of vertices which does not belong to any of the γ_t–sets of G with $|A| = p – k$. If A is empty, then $\gamma_{agt}(G) = \gamma_t(G)$. Now

$$\gamma_{agt}(G) = \frac{1}{|V(G)|}\sum_{v \in V(G)}\gamma_{vt}(G) = \frac{1}{p}\sum_{v_i \in S_i}\gamma_{vt}(G) + \frac{1}{p}\sum_{v_i \notin S_i}\gamma_{vt}(G) = \frac{1}{p}[k(|S_i|) + |A|(|S_i| + 1)]$$

$$= \frac{1}{p}[k(|S_i|) + (p – k)(|S_i| + 1)] = |S_i| + \frac{p-k}{p} < |S_i| + 1, \text{ since } p – k < p.$$

Hence $\gamma_{agt}(G) < \gamma_t(G) + 1$.

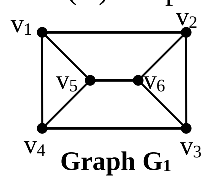
Theorem 3.7. If $\Delta(G) = p – 1$, then $\gamma_{agt}(G) = \gamma_t(G) = 2$.

Proof. Let G be a connected graph with $\Delta(G) = p – 1$, then $\gamma(G) = 1$. Let S = {u} be the dominating set of G where u is the vertex of degree p – 1. For a total dominating set there should not be an isolated vertex. Therefore $S_1 = S \cup \{v\}$ for any $v \in V(G)$ will be a total dominating set of G. Hence $\gamma_t(G) = 2$. This proves the first part of the theorem.

By the definition of an average total dominating set, $\gamma_{agt}(G) = \frac{1}{|V(G)|}\sum_{v \in V(G)}\gamma_{vt}(G)$. From the first part, $\gamma_{agt}(G) = \frac{1}{p}(2 + 2 + \ldots + p \text{ times}) = 2$. Hence the theorem.

Proposition 3.8. If $G \cong K_p$ ($p \geq 3$), then $\gamma_{agt}(G) = \gamma_t(G) = 2$.

Remark 3.9. The converse of the Theorem 3.3. is not true. That is, if there exist a graph G with $\gamma_t(G) = 2$, then it is not necessary that $\Delta(G) = p - 1$. Consider the cubic graph G_1 given in Figure 3.10.



Graph G_1

Figure 3.10.

The $S = \{v_5, v_6\}$ is a total dominating set. Therefore $\gamma_t(G) = 2$. But the graph is a cubic graph. Therefore $\Delta(G) = 3 \neq p - 1$.

The lower and upper bound of the average total domination number is obtained in the following Theorem.

Theorem 3.11. For any connected graph G, $2 \leq \gamma_{agt}(G) < \gamma_t(G) + 1$.

Proof. The proof of this theorem is an immediate consequent of Theorem 3.6. and Theorem 3.7.

Theorem 3.12. If there exists exactly one γ_t–set for a connected graph G, then $\gamma_{agt}(G) = \gamma_t(G) + 1 - \frac{\gamma_t(G)}{p}$ where $p = |V(G)|$.

Proof. Let S be the only γ_t – set of G. Then for |S| vertices, $\gamma_{vt}(G) = \gamma_t(G)$. By Theorem 3.5, for the remaining p - |S| vertices $\gamma_{vt}(G) = \gamma_t(G) + 1$. By the definition, $\gamma_{agt}(G) = \frac{1}{|V(G)|}\sum_{v \in V(G)} \gamma_{vt}(G) =$

$$\frac{1}{|V(G)|}\left(\sum_{v \in S} \gamma_{vt}(G) + \sum_{v \notin S} \gamma_{vt}(G)\right) = \frac{1}{p}(|S| \times \gamma_t(G) + (p - |S|) \times (\gamma_t(G) + 1)) = \frac{1}{p}(p \times \gamma_t(G) + p - |S|)$$

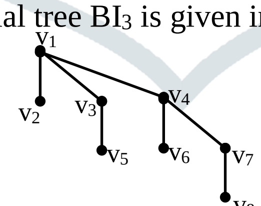
$= \gamma_t(G) + \frac{p - \gamma_t(G)}{p} = \gamma_t(G) + 1 - \frac{\gamma_t(G)}{p}$. Hence the theorem.

Definition 3.13. A **binomial tree** of order $k \geq 0$ with root R is the tree BI_k defined as follows.

(i) If k = 0, then $BI_k = B_0 = \{R\}$. That is, the binomial tree of order zero consists of a single vertex R.

(ii) If $k \geq 0$ then $BI_k = \{R, B_0, B_1, \ldots, B_{k-1}\}$. That is, the binomial tree of order $k \geq 0$ comprises the root and k binomial subtrees $B_0, B_1, \ldots, B_{k-1}$.

Illustration 3.14. The Binomial tree BI_3 is given in Figure 3.15.



BI_3

Figure 3.15.

The set $S = \{v_1, v_3, v_4, v_7\}$ is the only minimal total dominating set of the Binomial tree BI_3. The minimal total dominating sets containing the vertices v_2, v_5, v_6 and v_8 are as follows:

$S_1 = S \cup \{v_2\}$; $S_2 = S \cup \{v_5\}$; $S_3 = S \cup \{v_6\}$ and $S_4 = S \cup \{v_8\}$. Hence |S| = 4 and $|S_i| = |S| + 1 = 5$ for i = 1, 2, 3, 4. Therefore $\gamma_{agt}(BI_3) = \frac{1}{|V(G)|}\sum_{v \in V(G)} \gamma_{vt}(G) = \frac{1}{8}(4 \times 4 + 4 \times 5) = \frac{1}{8}(8 \times 4 + 4) = 4 + \frac{4}{8}$

$= 4 + \frac{1}{2} = \frac{9}{2} = \frac{p+1}{2}$.

Theorem 3.16. If G is a graph in which each vertex is either a support or a pendant vertex, then $\gamma_t(G) = s$ and $\gamma_{agt}(G) = s + 1 - \frac{s}{p}$ where s is the number of supports of G.

Proof. Let s be the number of supports of G. For a dominating set S, all the supports must be included in S. Hence $\gamma_t(G) = |S| = s$. This is the only γ_t – set of G. By Theorem 3.12, $\gamma_{agt}(G) = \gamma_t(G) + 1 - \frac{\gamma_t(G)}{p} = s + 1 - \frac{s}{p}$.

Corollary 3.17. If G is a graph in which each vertex is either a support or a pendant vertex and the number of supports are equal to the number pendant vertices, then $\gamma_t(G) = \frac{p}{2}$ and $\gamma_{agt}(G) = \frac{p+1}{2}$ where s is the number of supports of G.

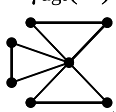
Proof. If in a graph G, the number of supports are equal to the number of pendant vertices then the number of supports are $\frac{p}{2}$. By Theorem 3.13, $\gamma_t(G) = \frac{p}{2}$ and $\gamma_{agt}(G) = s + 1 - \frac{s}{p} = \frac{p}{2} + 1 - \frac{p/2}{p} = \frac{p+1}{2}$.

Theorem 3.18. For a connected graph G if all the vertices are included in any one of the γ_t–sets of G, then $\gamma_{agt}(G) = \gamma_t(G)$.

Proof. Let V(G) be the vertex set of the connected graph G and |V(G)| = p. Let all the vertices of G are included in any one of the γ_t–sets of G. Then for any $v \in V(G)$, $\gamma_{vt}(G) = \gamma_t(G)$. By the definition of average total domination number $\gamma_{agt}(G) = \frac{1}{|V(G)|}\sum_{v \in V(G)} \gamma_{vt}(G) = \frac{1}{p} \times (\gamma_t(G) + \gamma_t(G) + \ldots + p \text{ times}) = \gamma_t(G)$. Hence the theorem.

Definition 3.19. **One point union cycle graph** $C_n(k)$ is the graph obtained by replacing k number of times the vertex of the complete graph K_1 by the cycle C_n. The vertex of K_1 is called the central vertex.

Example 3.20. Let G_1 be the one point union cycle $C_3(3)$. Then the vertex of K_1 will form a dominating set of G_1 and any one vertex together with that vertex will form the total dominating set of G_1. Hence $\gamma_{agt}(G) = \gamma_t(G) = 2$.



$G_1 \cong C_3(3)$

Figure 3.21.

Definition 3.22. A tree in which one vertex (called the **root**) is distinguished from all the others is called a **rooted tree**. In a rooted tree, the level (**depth**) of a vertex v is the length of the unique path from the root to v. Hence the root is at level 0. The **height** of a rooted tree is the length of a longest path from the root. If the vertex u immediately precedes the vertex v on the path from the root to v, then u is the **parent** of v and v is a **child** of u. Vertices having the same parent are called **siblings**. A vertex v is said to be **descendant** of a vertex u (u is then said to be an **ancestor** of v), if u is on the unique path from the root to v. If in addition, $v \neq u$, then v is a **proper descendant** of u (u is a **proper ancestor** of v). A **leaf** in a rooted tree is any vertex having no children. An **internal** vertex of a rooted tree is any vertex that is not a leaf.

Definition 3.23. An **m–ary** tree ($m \geq 2$) is a rooted tree in which each vertex has less than or equal to m children. When m = 2 the corresponding m–ary trees are called as **binary tree**. The total number of vertices in a complete binary tree of depth k is $2^k - 1$. A **complete m–ary**

tree is an m–ary tree in which each internal vertex has exactly m children and all leaves have the same depth.

Example 3.24:

Complete Binary Tree B_4

Figure 3.25.

Theorem 3.26. For a Complete Binary Tree B_k, the recursion formula for the total domination number is given by, $\gamma_t(B_k) = 2^{k-1} + 2^{k-2} + \gamma_t(B_{k-4})$ where k ($k \geq 8$ and $k \equiv 0, 1, 2$ or 3 (mod 4)) is the depth of the complete binary tree with the initial conditions

$\gamma_t(B_4) = 2^3 + 2^2 + 2^0 + 1$; $\gamma_t(B_5) = 2^4 + 2^3 + 2^0 + 1$; $\gamma_t(B_6) = 2^5 + 2^4 + 2^1 + 2^0$;

$\gamma_t(B_7) = 2^6 + 2^5 + 2^2 + 2^1$.

Proof. Let B_k denote a complete binary tree formed at the depth k. Let S_k be a γ_t–set of B_k

For a total dominating set there should not be any isolated vertex. Hence if the set of vertices in the i^{th} Level are included in S then the set of vertices in the $(i-1)^{th}$ Level should also be included in S. It is also found that the number of set of vertices in the i^{th} Level is twice than that of vertices in the $(i-1)^{th}$ Level. Also the pendant vertices are at the last Level. Therefore for a total dominating set the vertices are included from the $(k-1)^{th}$ Level that is, the backward inclusion method is applied.

Step 1. To calculate $\gamma_t(B_4)$**.**

The total dominating set S_4 include the vertices at the 3^{rd} Level and 2^{nd} Level. The vertices at the 4^{th} Level are dominated by the vertices at the 3^{rd} Level and the vertices at the 1^{st} Level are dominated by the vertices at the 2^{nd} Level. Now the vertex at the 0^{th} Level (that is, the root) is not dominated by any vertex and therefore it should be included in S_4. In addition to the root for a total dominating set any one vertex at the 1^{st} Level must also be included in S_4. It is observed that the set S_4 include the minimum number of vertices. Therefore total number of vertices in S_4 are $2^3 + 2^2 + 2^0 + 1$. Hence $\gamma_t(B_4) = 2^3 + 2^2 + 2^0 + 1$.

Step 2. To calculate $\gamma_t(B_5)$**.**

The total dominating set S_5 include the vertices at the 4^{th} Level and 3^{rd} Level. The vertices at the 5^{th} Level are dominated by the vertices at the 4^{th} Level and the vertices at the 2^{nd} Level are dominated by the vertices at the 3^{rd} Level. To dominate the vertices at the 1^{st} Level the vertex at the 0^{th} Level (that is, the root) is included in S_5. In addition to the root for a total dominating set any one vertex at the 1^{st} Level must also be included in S_5. It is observed that the set S_5 include the minimum number of vertices. Therefore total number of vertices in S_5 are $2^4 + 2^3 + 2^0 + 1$. Hence $\gamma_t(B_5) = 2^4 + 2^3 + 2^0 + 1$.

Step 3. To calculate $\gamma_t(B_6)$**.**

The total dominating set S_6 include the vertices at the 5^{th} Level and 4^{th} Level. The vertices at the 6^{th} Level are dominated by the vertices at the 5^{th} Level and the vertices at the 3^{rd} Level are dominated by the vertices at the 4^{th} Level. Now the vertex at the 2^{nd} Level is not dominated by any vertex and therefore the vertices at the 1^{st} should be included in S_6. In addition to this for a total dominating set the root must also be included in S_6. It is observed that the set S_6 include the minimum number of vertices. Therefore total number of vertices in S_6 are $2^5 + 2^4 + 2^1 + 2^0$. Hence $\gamma_t(B_6) = 2^5 + 2^4 + 2^1 + 2^0$.

Step 4. To calculate $\gamma_t(B_7)$**.**

The total dominating set S_7 include the vertices at the 6^{th} Level and 5^{th} Level. The vertices at the 7^{th} Level are dominated by the vertices at the 6^{th} Level and the vertices at the 4^{th} Level are dominated by the vertices at the 5^{th} Level. Now the vertex at the 3^{rd} Level is not dominated by any vertex and therefore the vertices at the 2^{nd} Level should be included in S_7. In addition to this for a total dominating set the vertices at the 1^{st} Level must also be included in S_7. The vertex at the 0^{th} Level (that is, the root) are dominated by the vertices at the 1^{st} Level. Therefore total number of vertices in S_7 are $2^6 + 2^5 + 2^2 + 2^1$. It is observed that the set S_7 include the minimum number of vertices. Hence $\gamma_t(B_7) = 2^6 + 2^5 + 2^2 + 2^1$.

This theorem is proved by the method of induction on k.

Case 1. $k \equiv 0 \pmod 4$

Let p = 4k. If k = 1 then by Step 1, $\gamma_t(B_4) = 2^3 + 2^2 + 2^0 + 1$. If k = 2 then proceeding by Step 1, $\gamma_t(B_8) = 2^7 + 2^6 + 2^3 + 2^2 + 2^0 + 1 = 2^{k-1} + 2^{k-2} + \gamma_t(B_4)$. If k = 3 then proceeding by Step 1, $\gamma_t(B_{12}) = 2^{11} + 2^{10} + 2^7 + 2^6 + 2^3 + 2^2 + 2^0 + 1 = 2^{k-1} + 2^{k-2} + \gamma_t(B_8)$. Proceeding in a similar manner, $\gamma_t(B_k) = 2^{k-1} + 2^{k-2} + \gamma_t(B_{k-4})$.

Case 2. $k \equiv 1 \pmod 4$

Let p = 4k + 1. If k = 1 then by Step 2, $\gamma_t(B_5) = 2^4 + 2^3 + 2^0 + 1$. If k = 2 then proceeding by Step 2, $\gamma_t(B_9) = 2^8 + 2^7 + 2^4 + 2^3 + 2^0 + 1 = 2^{k-1} + 2^{k-2} + \gamma_t(B_5)$. If k = 3 then proceeding by Step 2, $\gamma_t(B_{13}) = 2^{12} + 2^{11} + 2^8 + 2^7 + 2^4 + 2^3 + 2^0 + 1 = 2^{k-1} + 2^{k-2} + \gamma_t(B_{13})$. Proceeding in a similar manner, $\gamma_t(B_k) = 2^{k-1} + 2^{k-2} + \gamma_t(B_{k-4})$.

Case 3. $k \equiv 2 \pmod 4$

Let p = 4k + 2. If k = 1 then by Step 3, $\gamma_t(B_6) = 2^5 + 2^4 + 2^1 + 2^0$. If k = 2 then proceeding by Step 1, $\gamma_t(B_{10}) = 2^9 + 2^8 + 2^5 + 2^4 + 2^1 + 2^0 = 2^{k-1} + 2^{k-2} + \gamma_t(B_6)$. If k = 3 then proceeding by Step 3, $\gamma_t(B_{14}) = 2^{13} + 2^{12} + 2^9 + 2^8 + 2^5 + 2^4 + 2^1 + 2^0 = 2^{k-1} + 2^{k-2} + \gamma_t(B_{10})$. Proceeding in a similar manner, $\gamma_t(B_k) = 2^{k-1} + 2^{k-2} + \gamma_t(B_{k-4})$.

Case 4. $k \equiv 3 \pmod 4$

Let p = 4k + 3. If k = 1 then by Step 4, $\gamma_t(B_7) = 2^6 + 2^5 + 2^2 + 2^1$. If k = 2 then proceeding by Step 1, $\gamma_t(B_{11}) = 2^{10} + 2^9 + 2^6 + 2^5 + 2^2 + 2^1 = 2^{k-1} + 2^{k-2} + \gamma_t(B_7)$. If k = 3 then proceeding by Step 3, $\gamma_t(B_{15}) = 2^{14} + 2^{13} + 2^{10} + 2^9 + 2^6 + 2^5 + 2^2 + 2^1 = 2^{k-1} + 2^{k-2} + \gamma_t(B_{11})$. Proceeding in a similar manner, $\gamma_t(B_k) = 2^{k-1} + 2^{k-2} + \gamma_t(B_{k-4})$.

Hence in all the cases the recursion formula for the total dominating set of Complete Binary Tree at the depth k is $\gamma_t(B_k) = 2^{k-1} + 2^{k-2} + \gamma_t(B_{k-4})$.

Remark 3.27. For a Complete Binary Tree B_k at the depth k,
$\gamma_t(B_1) = 2^1 = 2$; $\gamma_t(B_2) = 2^1 + 2^0 = 3$; $\gamma_t(B_3) = 2^2 + 2^1 = 6$.

Note 3.28. The total dominating set S_k defined in Theorem 3.26. is the only minimum total dominating set of the Complete Binary Tree B_k.

Theorem 3.29. For a Complete Binary Tree at the depth k B_k, $\gamma_{agt}(B_k) = \gamma_t(B_k) + 1 - \dfrac{\gamma_t(B_k)}{2^k - 1}$.

Proof. By Theorem 3.12, for a connected graph G if there exists exactly one γ_t–set S for a G, then $\gamma_{agt}(G) = \gamma_t(G) + 1 - \dfrac{\gamma_t(G)}{p}$. Therefore $\gamma_{agt}(B_k) = \gamma_t(B_k) + 1 - \dfrac{\gamma_t(B_k)}{2^k - 1}$.

Example 3.30. Consider the Figure 3.25. given in Example 3.24. which is a Complete Binary Tree B_4. By the Theorem 3.26, the total domination number is $\gamma_t(B_4) = 2^3 + 2^2 + 2^0 + 1 = 14$. By the Theorem 3.29, the average total domination number is $\gamma_{agt}(B_4) = \gamma_t(B_4) + 1 - \dfrac{\gamma_t(B_4)}{p} = 14 + 1 - \dfrac{14}{31}$.

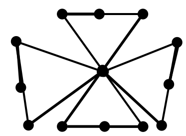
Proposition 3.31. For a Complete Ternary Tree T_k, the recursion formula for the total domination number is given by, $\gamma_t(T_k) = 3^{k-1} + 3^{k-2} + \gamma_t(B_{k-4})$ where k ($k \geq 8$ and $k \equiv 0, 1, 2$ or $3 \pmod 4$) is the depth of the complete ternary tree with the initial conditions
$\gamma_t(T_4) = 3^3 + 3^2 + 3^0 + 1$; $\gamma_t(T_5) = 3^4 + 3^3 + 3^0 + 1$; $\gamma_t(T_6) = 3^5 + 3^4 + 3^1 + 3^0$;
$\gamma_t(B_7) = 3^6 + 3^5 + 3^2 + 3^1$.

Proposition 3.32. For a Complete Ternary Tree at the depth k T_k, $\gamma_{agt}(T_k) = \gamma_t(T_k) + 1 - \dfrac{\gamma_t(T_k)}{3^k - 1}$.

Theorem 3.33. For any integer k, there exist a graph G with $\gamma_{agt}(G) = \gamma_t(G) = k + 1$ where $k \geq 2$.

Proof. For the cycle C_4, $\gamma_t(C_4) = 2$, since any two adjacent vertices will form a total dominating set. Consider the one - point union cycle $C_4(k)$ where k is an integer, $k \geq 2$. It follows that the central vertex and any vertex adjacent to the central vertex will form a total dominating set of $C_4(k)$. Also, there are k copies of C_4 in $C_4(k)$. Hence $\gamma_t(C_4(k)) = k + 1$. Since all the vertices in $C_4(k)$ can be included in any one of the γ_t – set of G, it follows that $\gamma_{agt}(G) = \gamma_t(G) = k + 1$ by Theorem 3.18.

Example 3.34. For the graph $G \cong C_4(4)$ given in Figure 3.35., $\gamma_{agt}(G) = \gamma_t(G) = 5$.



$C_4(4)$

Figure 3.35.

Definition 3.36. Let G be a connected graph. The **Thorn Graph** G* of the graph G is obtained by attaching a pendant edge to all the vertices of G.

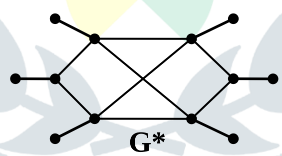
Theorem 3.37. Their exist a connected graph G* with $\gamma_t(G^*) = \frac{p}{2}$ and $\gamma_{agt}(G^*) = \frac{p+1}{2}$ where p is the number of vertices of G*.

Proof. Let G be a connected graph with p/2 vertices and G* be the thorn graph of G. Then $|V(G^*)| = p$. Then by the definition of the thorn graph there are pendant vertices attached to each vertex of G. For a dominating set S of G*, all the supports are to be included that is, the vertices of G are to be included in S. Since G is connected S will be a total dominating set of G*. Therefore $\gamma_t(G^*) = |S| = |V(G)| = \frac{p}{2}$.

Also, S is the only total dominating set of G*. Hence by Theorem 3.18, $\gamma_{agt}(G^*) = \gamma_t(G^*) + 1 - \frac{\gamma_t(G^*)}{p}$

$$= \frac{p}{2} + 1 - \frac{p/2}{p} = \frac{p+1}{2}.$$

Example 3.38. For the graph G* given in Figure 3.39, $\gamma_t(G^*) = 6$ and $\gamma_{agt}(G^*) = \frac{13}{2}$.



G*

Figure 3.39.

References

[1] Aslan E and Kirlangic, *The average lower domination number of graphs,* Bulletin of the International Mathematical Virtual Institute, Vol. 3(2013), p. 155 – 160.

[2] Aslan E, *The average lower connectivity of graphs,* Research Article, Hindawi Publishing Corporation, Journal of Applied Mathematics, Vol. 2014, p. 1 – 4.

[3] Cockayne E, Dawes R, Hedetniemi S, *Total domination in graphs,* Networks Vol. 10(1980), p. 211 – 219.

[4] Dogan D, *Average lower domination number for some middle graphs,* International J. Math. Combin. Vol. 4(2012), p. 58 – 67.

[5] Henning M A, *Trees with equal average domination and independent domination numbers,* Ars Combin., 71(2004), p. 305 – 318.

[6] Muthammai S, Meenal N, *Average co–isolated locating domination number on trees*, Elixir Applied Mathematics, Vol. 92(2016), p. 38858 – 38863, ISSN: 2229 – 712X.

[7] Ore O, *Theory of Graphs*, Amer. Math. Soc. Colloq. Publ. 38, Providence, RI, 1962.