# Existence of best proximity points for cyclic B-contraction mappings

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Abstract

In this paper, we obtain best proximity points for cyclic B-contraction mappings that unify and extend results of Anthony Eldred et al, Bright et al and many authors.

Keywords:  Cyclic B-contraction, Proximity point, Fixed points.

## 1. Introduction and preliminaries

Anthony Eldred et al[1] introduced the concept of cyclic contraction mappings and obtained best proximity points for cyclic contractions. Bright et al[2] proved xed point theorems by introducing the new concept B-contraction mapping. In this paper, we introduce the new concept cyclic B-contraction and and obtain best proximity point theorems. Our results unify and extend all the results of Anthony Eldred et al,Bright et al and many authors. The following notions are used subsequently:

$$d(A; B) := \text{inff} d(x; y) : x \in A \text{ and } y \in Bg;$$

$$A_0 = fx \in A : d(x; y) = d(A; B) \text{ for some } y \in Bg;$$

$$B_0 = fy \in B : d(x; y) = d(A; B) \text{ for some } x \in Ag:$$

A map $T : A \to B$ is said to have proximity point if there exists $x \in A$ such that $d(x; T x) = d(A; B):$

De nition 1.1. Let $(X; d)$ be a metric space $X; S; T : X \to X$ be a selfmaps. Then $S; T$ have a common best proximity point if there exists $x \in X$ such that $d(x; T x) = d(x; Sx) = d(A; B):$

De nition 1.2. [1] A subset K of a metric space X is said to be boundedly compact if each bounded sequence in K has a subsequence converging to a point in K:

Suppose X is uniformly convex(and hence re exive)Banach space with modulus of convexity : Then (" > 0) for " > 0; and (:) is strictly increasing. Moreover if $x; y; p \in X; R > 0$ and $r \in [0; 2R];$

$$k \quad x \quad yk \quad r = \quad ) k \frac{x \quad p}{2} \quad R \quad k \qquad R$$
$$k \qquad k \qquad ;$$

De nition 1.3. [1] Let $A; B$ be nonempty subsets of a metric space $X; T : A \cup B \to A \cup B$ is said to be cyclic contraction if

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(1)  T (A)   B and T (B)   A

(2)  d(T x; T y)   kd(x; y) + (1  k)d(A; B) for some k 2 (0; 1); for all x 2 A; y 2 B:

De nition 1.4. Let T : X ! X, where (X; d) is a complete metric space is called a B-contraction if there exists positive real number ; ; such that 0 + 2 + 2 < 1 for all x; y 2 X the following inequality holds:

$$d(T x; T y)   d(x; y) +  [d(x; T x) + d(y; T y)] +  [d(x; T y) + d(y; T x)] \qquad (1)$$

Lemma 1.5. [1] Let A be a nonempty closed and convex subset, B be a nonempty closed subset of a uniformly convex Banach space and fx$_n$g; fz$_n$g be sequences in A and fy$_n$g be a sequence in B satisfying:

(1)  kz$_n$   y$_n$k ! d(A; B)

(2)  for every " > 0; there exists n$_0$ 2 N such that for all m > n   n$_0$; kx$_m$   y$_n$j   d(A; B) + ":

Then for every " > 0 there exists n$_1$ 2 N such that for all m > n     n$_1$; kx$_m$   z$_n$j    ":

Lemma 1.6. [1] Let A be a nonempty closed and convex subset, B be a nonempty closed subset of a uniformly convex Banach space and fx$_n$g; fz$_n$g be sequences in A and fy$_n$g be a sequence in B satisfying:

(1)  kx$_n$   y$_n$k ! d(A; B)

(2)  kz$_n$   y$_n$k ! d(A; B):

Then kx$_n$   z$_n$k ! 0:

2. Proximity point theorems for cyclic B-contractions

De nition 2.1. Let A; B be nonempty closed subsets of a metric space X; T : A [ B ! A [ B is said to be cyclic B-contraction if

(1)  T is cyclic and

(2)  there exists nonnegative real numbers ; ; with + 2 + 2 < 1 such that d(T x; T y) d(x; y) + [d(x; T x) + d(y; T y)] + [d(x; T y) + d(y; T x)] + [1 ( + 2 + 2 )]d(A; B)

for all x; y 2 A [ B and     + 2   + 2   < 1:

Proposition 2.2. Let A; B be nonempty subsets of a metric space X: Suppose T : A[B ! A[B is a cyclic B- contraction map. For any x$_0$ 2 A [ B; let x$_{n+1}$ = T x$_n$ for all n 2 N [ f0g:
Then d(x$_n$; x$_{n+1}$) ! d(A; B) as n ! 1:

Proof. Let x$_0$ 2 A [ B: Then

d(x$_n$; x$_{n+1}$)  =  d(T x$_{n 1}$; T x$_n$)
        d(x$_{n 1}$; x$_n$) +  [d(x$_{n 1}$; T x$_{n 1}$) + d(x$_n$; T x$_n$)] +
                        [d(x$_{n 1}$; T x$_n$) + d(x$_n$; T x$_{n 1}$)] + [1 ( + 2 + 2 )]d(A; B) d(x$_{n 1}$; x$_n$) +  [d(x$_{n 1}$; x$_n$) + d(x$_n$; x$_{n+1}$)] +
                        [d(x$_{n 1}$; x$_n$) + d(x$_n$; x$_{n+1}$)] + [1 ( + 2 + 2 )]d(A; B)
        ( + + )d(x$_{n 1}$; x$_n$) + ( + )d(x$_n$; x$_{n+1}$)] + [1 ( + 2 + 2 )]d(A; B):

Then, we have

$$[1 \ ( \quad + \ )]d(x_n; x_{n+1}) \quad ( \quad + \quad + \ )d(x_{n\ 1}; x_n) + [1 \ ( \quad + 2 \quad + 2 \ )]d(A; B):$$

Which gives as

$$d(x_n; x_{n+1}) \quad \frac{+ \ +}{[1 \ ( \ + \ )]}d(x_{n\ 1}; x_n) + \frac{1 \ ( \ + 2 \ + 2 \ )}{[1 \ ( \ + \ )]} d(A; B)$$

$$= [1 \ ( \ + \ )] \quad d(x_{n\ 1}; x_n) + f1 \quad [1 \ ( \ + \ )] \quad gd(A; B):$$

Now,

$$d(x_n; x_{n+1}) \quad \frac{+ \ +}{ } \quad^2 \quad d(x_{n\ 2}; x_{n\ 1}) + f1 \quad \frac{2 \ + 2}{ }\quad^2 \quad gd(A; B):$$

Continuing this process, we get

$$d(x_n; x_{n+1}) \quad [1 \quad \frac{+}{+}( \ + \ )] \quad^n \quad d(x_0; x_1) + f1 \quad [1 \ ( \ + \ )]^n \quad gd(A; B):$$

Since $+ 2 \ + 2 \ < 1;$ $\frac{+ \ +}{[1 \ ( \ + \ )]} < 1:$ Hence $d(x_n; x_{n+1}) \to d(A; B)$ sa $n \to 1:$

**Corollary 2.1.** Let $T : X \to X$, where $(X; d)$ is a complete metric space, be a B-contraction. Then T has a unique fixed point.

**Proof.** The proof of the corollary follows by putting $A = B = X:$ $\square$

**Corollary 2.2.** Let $A; B$ be nonempty subsets of a metric space $X:$ Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction map. Then starting with any $x_0 \in A \cup B;$ we have $d(x_n; x_{n+1}) \to d(A; B)$ where $x_{n+1} = T x_n; n = 0; 1; 2; \dots$

**Proof.** The proof of the corollary follows by putting $= = 0:$ $\square$

**Proposition 2.3.** Let $A; B$ be nonempty closed subsets of a complete metric space $X; T : A \cup B \to A \cup B$ be a cyclic B-contraction map, let $x_0 \in A$ and define $x_{n+1} = T x_n:$ Suppose $fx_{2n}g$ has a convergent subsequence in $A:$ Then there exists $x \in A$ such that $d(x; T x) = d(A; B):$

**Proof.** Let $fx_{2n_k}g$ be a subsequence of $fx_{2n}g$ and $\lim_{k \to 1} x_{2n_k} = x$ for some $x \in A:$ Now,

$$d(A; B) \quad d(x; x_{2n_k\ 1}) \quad d(x; x_{2n_k}) + d(x_{2n_k}; x_{2n_k\ 1}):$$

Taking limit as $n \to 1$ in the above inequality, we have

$$d(x; x_{2n_k\ 1}) \to d(A; B):$$

Since

$$d(A; B) \quad d(x_{2n_k}; T x) \quad d(x_{2n_k\ 1}; x):$$

As $n \to 1;$ we have

$$d(x; T x) = d(A; B):$$

$\square$

**Proposition 2.4.** Let $A; B$ be nonempty subsets of a metric space $X; T : A \cup B \to A \cup B$ be a cyclic B-contraction map. Then for any $x_0 \in A$ and $x_{n+1} = T x_n;$ the sequences $fx_{2n}g$ and $fx_{2n+1}g$ are bounded.

Proof. Suppose $x_0 \in A$(the proof is similar when $x_0 \in B$), then by Proposition(2.1), $d(x_{2n}; x_{2n+1}) \to$ $d(A; B)$ as $n \to 1$: Hence it is enough to prove $\{x_{2n+1}\}$ is bounded. Let $r = \dfrac{^{+\ +}}{[1\ \ (+)]}^{2}$ : Suppose $\{x_{2n+1}\}$ is not bounded. Then there exists $n_0 \in N$ such that

$$d(x_2; x_{2n0+1}) > M \text{ and } d(x_2; x_{2n0\ 1})\ M$$

where $M > \max\left\{\dfrac{n\ _{r^2\ 1}}{\frac{d(x_0; x_2)}{1}} + d(A; B); d(x_1; x_{20})\right\}$ : By the cyclic B-contraction property of $T$; we have

$$M < d(x_2; x_{2n0+1}) = d(T^2 x_0; T^2 x_{2n0\ 1})$$
$$r^2 d(x_0; x_{2n0\ 1}) + \{1\ r^2\}d(A; B)$$

. Hence

$$\frac{M\ \ d(A; B)}{r^2} + d(A; B) < d(x_0; 2x_{2n0\ 1})\quad d(x_0; x_2) + d(x_2; x_{2n0\ 1})\quad d(x_0; x_2) + M:$$

Thus

$$M < \frac{d(x_0; x_2)}{\frac{1}{r^2\ 1}} + d(A; B):$$

Which is a contradiction. Hence $\{x_{2n+1}\}$ is bounded.　　□

Theorem 2.5. Let $A; B$ be nonempty closed subsets of a metric space $X$ and $T : A \cup B \to A \cup B$ is a cyclic B contraction. If either $A$ or $B$ is boundedly compact, then there exists $x \in A \cup B$ such that $d(x; Tx) = d(A; B)$:

Proof. It follows directly from propositions (2.3) and (2.4).　　□

Corollary 2.3. Let $A; B$ be nonempty closed subsets of a metric space $X$ and $T : A \cup B \to A \cup B$ is a cyclic contraction. If either $A$ or $B$ is boundedly compact, then there exists $x \in A \cup B$ such that $d(x; Tx) = d(A; B)$:

3.　Proximity point theorems for cyclic B-contractions on uniformly convex Banach space

Theorem 3.1. Let $A; B$ be nonempty closed and convex subsets of a uniformly convex Banach space $X$: Suppose $T : A \cup B \to A \cup B$ is a cyclic B-contraction map, then there exists a unique best proximity point $x \in A$: Further if $x_0 \in A$ and $x_{n+1} = T x_n$; then $\{x_{2n}\}$ converges to the best proximity point.

Proof. Since $\|x_{2n}\ T x_{2n}\| \to d(A; B)$ and $\|T^2 x_{2n}\ T x_{2n}\| \to d(A; B)$: By Lemma(1.6), $\|x_{2n}\ T^2 x_{2n}\| \to 0$: Similarly we can show that $\|T x_{2n}\ T x_{2(n+1)}\| \to 0$: We now show that for every $\varepsilon > 0$; there exists $n_0 \in N$ such that for all $m > n\ \ n_0$; $\|x_{2m}\ T x_{2n}\|\ d(A; B) + \varepsilon$: Suppose not, then there exists $\varepsilon > 0$ such that for all $k \in N$; there exists $m_k > n_k\ \ k$ for which

$$\|x_{2m_k}\ T x_{2n_k}\|\ d(A; B) + \varepsilon:$$

$$d(A; B) + \varepsilon\ \ \|x_{2m_k}\ T x_{2n_k}\|$$
$$\|x_{2m_k}\ x_{2(m_k+1)}\| + \|x_{2(m_k+1)}\ T x_{2n_k}\|:$$

Hence $\lim_{k!1} kx_{2m_k}$      $T x_{2n_k} k = d(A; B) + "$: Now

$$kx_{2m_k} \quad T x_{2n_k} k \quad kx_{2m_k} \quad T x_{2(m_k+1)} k + kT x_{2(m_k+1)} \quad T x_{2(n_k+1)} k + kT x_{2(n_k+1)} \quad T x_{2n_k} k$$

and
$kI x_{2(m_k+1)} \quad I x_{2(n_k+1)}K$

$$[1 \quad ( + )]! \quad 2 \qquad (^1 \quad [1 \qquad 2 \quad d(A; B):$$
$$+ + \qquad\qquad\qquad +$$

Thus
$d(A; B) + "$

$$[1 \qquad _2(d(A; B) + ") + (_1 \qquad _2)d(A; B)$$
$$+ \qquad\qquad\qquad\qquad + +$$

$$+$$
$$= d(A; B) + \qquad [1 \quad ( + )]! \quad _2$$
$$+$$

$$< \ d(A; B) + "$$

which is a contradiction.Therefore $fx_{2n}g$ is a Cauchy sequence by Lemma(1.5) and hence con-verges to some $x \ 2 \ A$: from proposition(2.3), we have $kx \ T \ xk = d(A; B)$: Now, we have to prove the uniqueness. Suppose $x$ and $y$ are the proximity points for $T$ and $x \ 6= \ y$; that is $kx \ T \ xk = d(A; B)$ and $ky \ T \ yk = d(A; B)$ where necessarily, $T^2x = x$ and $T^2y = y$: Therefore

$$kT x \quad yk = kT x \quad T^2yk \quad kx \quad T yk; kT y \quad xk = kT y \quad T^2xk \quad ky \quad T xk;$$

which implies $kT y \quad xk = ky \quad T xk$: But since $ky \quad T xk > d(A; B)$; it follows that $kT y \quad xk < ky \quad T xk$; which is a contradiction. Hence $x = y$:

Corollary 3.1. Let $A; B$ be nonempty closed and convex subsets of a uniformly convex Banach space $X$: Suppose $T : A [ B ! A [ B$ is a cyclic contraction map, then there exists a unique best proximity point $x \ 2 \ A$: Further if $x_0 \ 2 \ A$ and $x_{n+1} = T \ x_n$; then $fx_{2n}g$ converges to the best proximity point.

Proof. The proof of the corollary follows by putting if $\quad = \quad = 0$:         □

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