Common Fixed Point under Weaker Condition of Compatibility in Metric spaces

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Abstract: In this paper we use the notion of E. A. Property in metric space and prove a common fixed point theorem for weakly compatible mappings also given example in support of our theorem.

Index Terms **- Fixed point, metric space, weakly compatible maps, E. A. Property.**

I. Introduction and Preliminaries

In 1976, Jungck ([1]) gave a generalization of the Banach's contraction theorem for a pair of self-mappings in a complete metric space (X, d) and perhaps he is the first who introduced three conditions at a time i.e., Commuting, continuous maps and containment of ranges in the history of fixed point theorem and applications.

After Jungck ([1]) in 1976, S. Sessa ([4]) in 1982 introduced the concept of weakly commuting maps by generalizing commuting maps. It is interesting to note that commuting maps are weakly commuting but the converse is generally not true.

Definition 1.1: Two mappings S and T defined on a metric space (X, d) into itself is said to be weakly commuting maps if and only if

$$d(STx, TSx) \leq d(Tx, Sx) \text{ for all } x \in X.$$

In 1986, Jungck ([3]) again proposed a generalization of the concept of weakly commuting mappings which is weaker than weakly commuting maps called compatible mappings.

In 1998, Jungck and Rhoades ([5]) introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but converse need not be true.

Definition 1.2: Let A and S be two self-mappings of a metric space (X, d) are say that A and S satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 1.3: A pair of maps A and S is called weakly compatible pair if they commute at coincidence points.

In this paper we use the notion of E. A. Property in metric space and prove a common fixed point theorem for weakly compatible mappings also given example in support of our theorem.

II. Main Results

Theorem 3.1: Let A, B, S and T be mappings from a metric space (X, d) into itself such that

(3.1) $A(X) \cup B(X) \subseteq S(X) \cap T(X)$,

(3.2) the pair {A, S} and {B, T} are weak compatible maps,

(3.3) $d(Ax, By) \leq \varphi(\max\{ d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\})$

(3.4) $S(X) \cap T(X)$ is a closed subspace of X.

(3.5) the pair {A, S} and {B, T} are satisfying the E. A. property,

Where $\varphi : [0, \infty) \to [0, \infty)$ is a non - decreasing and upper semi - continuous function and $\varphi(t) < t$ for all $t > 0$ Then A, B, S and T have a unique common fixed point in X.

Proof: Since {A, S} and {B, T} are satisfy the E. A. property so there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = p$$

Since $A(X) \cup B(X) \subseteq S(X) \cap T(X)$ and $S(X) \cap T(X)$ is closed subspace of X, so $A(X) \subset T(X)$ and $B(X) \subset S(X)$, then there exists u, v in X such that $Su = p$ and $Tv = t$

Now, we shall prove that $Au = Su$.

By using condition (3.3), we have

$$d(Au, By_n) \leq \varphi(\max\{ d(Su, Ty_n), d(Su, Au), d(Ty_n, By_n), \tfrac{1}{2}[d(Su, By_n) + d(Ty_n, Au)]\})$$

as $n \to \infty$

$$d(Au, p) \leq \varphi(\max\{ d(Su, p), d(Su, Au), d(p, p), \tfrac{1}{2}[d(Su, p) + d(p, Au)]\})$$

$$d(Au, t) \leq \varphi(\max\{ d(t, p), d(t, Au), 0), \tfrac{1}{2}[d(t, p) + d(p, Au)]\})$$

Since $Su = p$, so

$$d(Au, p) \leq \varphi(\max\{ d(p, p), d(p, Au), 0), \tfrac{1}{2}[d(p, p) + d(p, Au)]\})$$

$$= \varphi(\max\{0, d(p, Au), 0), \tfrac{1}{2}d(p, Au)\})$$

$= \varphi(d(Au, p)) < (Au, p)$ a contradiction.
Which means that Au = p and so Au = Su = p
Now, we shall show prove that Tv = Bv
Again by condition (3.3), we have
$d(Ax_n, Bv) \leq \varphi(\max\{ d(Sx_n, Tv), d(Sx_n, Ax_n), d(Tv, Bv), \frac{1}{2}[d(Sx_n, Bv) + d(Tv, Ax_n)]\})$
as $n \rightarrow \infty$
$d(t, Bv) \leq \varphi(\max\{ d(t, Tv), d(t, t), d(Tv, Bv), \frac{1}{2}[d(t, Bv) + d(Tv, t)]\})$
Since Tv = t, so
$d(t, Bv) \leq \varphi(\max\{d(t, t), 0, d(t, Bv),), \frac{1}{2}[d(t, Bv) + d(t, t)$
$= \varphi(\max\{0, 0, d(t, Bv), \frac{1}{2}[d(t, Bv) + 0]\})$
$= \varphi(\max\{d(t, Bv), \frac{1}{2}d(t, Bv)\}$
$= \varphi(d(t, Bv)) < d(t, Bv)$ this is a contradiction.
Which means that t = Bv and Tv = Bv = t
Now we have to prove that t = p if not i.e., $t \neq p$ then by condition (3.3), we get
$d(t, p) = d(Ax_n, By_n) \leq \varphi(\max\{ d(Sx_n, Ty_n), d(Sx_n, Ax_n), d(Ty_n, By_n), \frac{1}{2}[d(Sx_n, By_n) + d(Ty_n, Ax_n)]\})$
as $n \rightarrow \infty$
$d(t, p) \leq \varphi(\max\{ d(t, p), d(t, t), d(p, p), \frac{1}{2}[d(t, p) + d(p, t)]\})$
$\leq \varphi(\max\{ d(t, p), 0, 0, d((t, p)\})$
Now
$d((t, p) \leq \varphi(\max(d(t, p))) < d(t, p)$
This is a contradiction.
Which means that t = p, so now we have
Au = Su = Tv = Bv = t.
Now, we shall assume the pair {A, S} is weak compatible maps, so
SAu = ASu $\Rightarrow$ St = At.
Similarly, Tt = Bt, by assuming {B, T} is weak compatible pair of maps.
Now, we shall that t is a common fixed point of A and S. Let if possible, $At \neq t$, then by again condition (3.3), we have
$d(At, Bv) \leq \varphi(\max\{ d(St, Tv), d(St, At), d(Tv, Bv), \frac{1}{2}[d(St, Bv) + d(Tv, At)]\})$
[Since Au = Su = Tv = Bv = t] so,
$d(At, t) \leq \varphi(\max\{ d(St, t), d(St, At), d(t, t), \frac{1}{2}[d(St, t) + d(t, At)]\})$
[Since At = St] so,
$d(At, t) \leq \varphi(\max\{ d(At, t), 0, 0, \frac{1}{2}[d(At, t) + d(t, At)]\})$
$= \varphi(\max\{ d(At, t), \frac{1}{2}[d(At, t) + d(t, At)]\})$
$= \varphi(d(At, t)) < d(At, t)$, a contradiction.
Which means that At = t and At = St = t.
Similarly, we can show that Bt = Tt = t.
Therefore, the mappings A, B, S, T have a common fixed point.
For uniqueness: suppose that there exists another common fixed point z for A, B, S and T such that $z \neq t$ then by (3.3) we have
$d(At, Bz) \leq \varphi(\max\{ d(St, Tz), d(St, At), d(Tz, Bz), \frac{1}{2}[d(St, Bz) + d(Tz, At)]\})$
$d(t, z) \leq \varphi(\max\{ d(t, z), d(t, t), d(z, z), \frac{1}{2}[d(t, z) + d(z, t)]\})$
$d(t, z) \leq \varphi(\max\{ d(t, z), 0, 0, d(t, z)\})$
$d(t, z) \leq \varphi(d(t, z)) < d(t, z)$ a contradiction.
Then z = t.
Hence A, B, S and T have a unique common fixed point in X.
If we put S =T, we get the following result.

Corollary: Let A, B and S be a mappings from a metric space (X, d) into itself such that
(1) $A(X) \cup B(X) \subseteq S(X)$,
(2) the pair {A, S} and {B, S} are weak compatible maps,
(3) $d(Ax, By) \leq \varphi(\max\{ d(Sx, Sy), d(Sx, Ax), d(Sy, By), \frac{1}{2}[d(Sx, By) + d(Sy, Ax)]\})$
Where : $[0, \infty) \rightarrow [0, \infty)$ is a non - decreasing and upper semi - continuous function and $\varphi(t) < t$ for all $t > 0$.
(4) S(X) is a complete subspace of X.
(5) the pair {A, S} and {B, S} are satisfying the E. A. property,
then A, B and S have a unique common fixed point in X.
Suppose A = B and S = T, we get the corollary.

Corollary: Let A and S be self – maps of a metric space (X, d) such that
(1) $A(X) \subseteq S(X)$,
(2) the pair {A, S} is weak compatible maps,
(3) $d(Ax, Ay) \leq \varphi(\max\{d(Sx, Sy), d(Sx, Ax), d(Sy, By), \frac{1}{2}[d(Sx, Ay) + d(Sy, Ax)]\})$
where $\varphi: [0, \infty) \to [0, \infty)$ is a non - decreasing and upper semi - continuous function and $\varphi(t) < t$ for all $t > 0$.
(4) S(X) is a complete subspace of X.
(5) the pair {A, S} is satisfying the E. A. property,
then A and S have a unique common fixed point in X.
If we put A = B = S = T. Then we have the following result.

Corollary: Let A be a self-map of a metric space (X, d) such that
(3) $d(Ax, Ay) \leq \varphi(\max\{d(Ax, Ay), d(Ax, Ax), d(Ay, Ay), \frac{1}{2}[d(Ax, Ay) + d(Ay, Ax)]\})$
Where $\varphi : [0, \infty) \to [0, \infty)$ is a non - decreasing and upper semi - continuous function and $\varphi(t) < t$ for all $t > 0$.
(4) A(X) is a complete subspace of X,
then A and S have a unique common fixed point in X.
Example: Let X = [0, ∞). Define A, S: X → X by $Ax = \frac{x}{4}$ and $Sx = \frac{3x}{4}$ $\forall x \in X$.
Consider the sequence $x_n = \frac{1}{n}$. Clearly $\lim_{n \to \infty} x_n = Ax_n = \lim_{n \to \infty} x_n = Sx_n = 0$.
Then S and A satisfy (E. A.) property.

Example: Let X = [2, ∞). Define A, S: X → X by Ax = x + 1 and Sx = 2x + 1,
$\forall x \in X$, suppose that the property (E. A.) holds, then there exists a sequence $\{x_n\}$ in X satisfying
$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z$ for some $z \in X$.
Therefore
$\lim_{n \to \infty} x_n = z - 1$ And $\lim_{n \to \infty} x_n = \frac{z-1}{2}$
Thus z = 1, which is a contradiction, since $1 \notin X$. Hence A and S don't satisfy
(E. A.) Property.

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