# Splines overview of Non-uniform rational B-spline – A Study

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**Abstract**

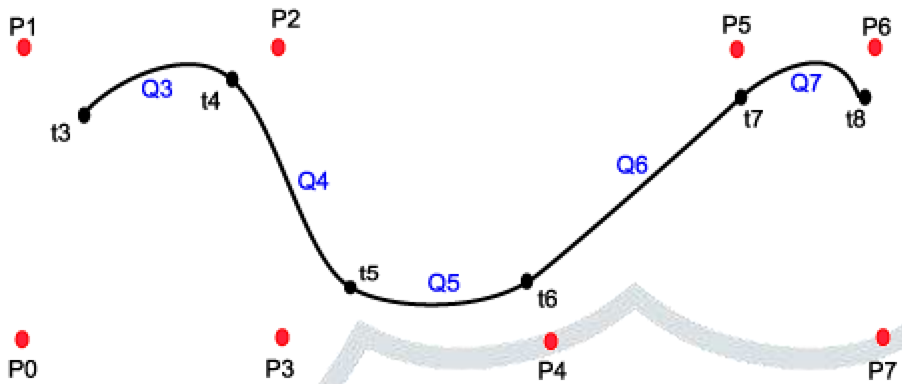
This paper attempts to study **Non-uniform rational** basis **spline** (**NURBS**) a mathematical model commonly used in computer graphics for generating and representing curves and surfaces. The shape of the surface is determined by control points. NURBS (nonuniform rational B-splines) are mathematical representations of 2- or 3-dimensional objects, which can be standard shapes (such as a cone) or free-form shapes (such as a car). NURBS are used in computer graphics and the CAD/CAM industry and have come to be regarded as a standard way to create and represent complex objects. In addition to curves and surfaces, NURBS can also represent hypersurfaces. Most sophisticated graphic creation tools provide an interface for using NURBS, which are flexible enough to design a wide range of shapes - anything from points to straight lines to conic sections. NURBS are compact expressions that can be evaluated and displayed quickly. NURBS work especially well in 3-D modeling, allowing the designer to easily manipulate control vertices, called ISO curves, and control curvature and the smoothness of contours. NURBS are defined by both control points and weights. It takes very little data to define a NURB. These variations which are referred to as generalized non-uniform rational B-splines (GNURBS) serve as an alternative interactive shape design tool, and provide improved approximation abilities in certain applications. GNURBS are obtained by decoupling the weights associated with control points along different physical coordinates. This unexplored idea brings the possibility of treating the weights as additional degrees of freedoms. It will be seen that this proposed concept effectively improves the capability of NURBS, and circumvents its deficiencies in special applications. Further, it is proven that these new representations are merely disguised forms of classic NURBS, guaranteeing a strong theoretical foundation, and facilitating their utilization. A few numerical examples are presented which demonstrate superior approximation results of GNURBS compared to NURBS in both cases of smooth and non-smooth fields. Finally, in order to better demonstrate the behavior and abilities of GNURBS in comparison to NURBS, an interactive MATLAB toolbox has been developed and introduced.

A spline is a usually curvy pattern used to guide someone shaping something large, such as a boat hull. The B-spline is based (the B stands for basis ) on four local functions or control points that lie outside the curve itself. Nonuniform is the idea that some sections of a defined shape (between any two points) can be shortened or elongated relative to other sections in the overall shape.

*Key words: Non-uniform rational basis spline - NURBS, GNURBS, B-spline, hypersurfaces.*

**Introduction**

Rational describes the ability to give more weight to some points in the shape than to other points in considering each positions relation to another object. (This is sometimes referred to as a 4th dimensional characteristic.) . We start out with a situation that may be characterized as Uniform Non Rational B-splines.




We have m+1=8 control points, and m-2=5 curve segments. The curve may be described by a parameter running from t3 to t8. A curve segment is limited by one t-step. The curve Qi is described by t running from ti to ti+1.

The segments are coupled according to this scheme:

- Q3 depends on P0 ,P1 ,P2 ,P3
- Q4 depends on P1 ,P2 ,P3 ,P4
- Q5 depends on P2 ,P3 ,P4 ,P5
- ...
- $Q_i$ depends on $P_{i-3}$ ,$P_{i-2}$ ,$P_{i-1}$ ,$P_i$

Each curve is described by 4 control points. This means that we have local control in the sense that no other points than those 4 influence the curves shape. In the same manner we see that each control point is involved i 4 curve segments.

we note that this is interesting compared to what we can achieve with Bezier curves. If we wanted a Bezier curve with the same amount of control points we would have to increase the level of the curve to m-1 and we would have a situation where all control points would influence the curve as such. If we make a curve by connecting cubical Bezier curves, we would have local control, but we would not control the continuity across segments when we change single control points.

we will look a little closer at the segments. We know from the modules *Bezier* and *Polynoms* the general form of a curve or curve segment:

$Q(t)=T \cdot M \cdot G$,

where T is a row vector describing the level of the curve, M is a 4x4 matrix which is special for the curve form and G is column vector describing the geometrical restrictions.

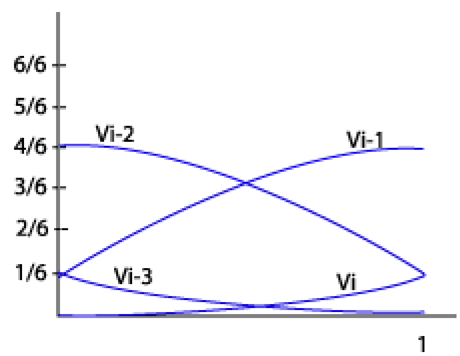
For curve segment i we can write:

$$T_i = \begin{bmatrix}(t-t_i)^3 & (t-t_i)^2 & (t-t_i) & 1\end{bmatrix}, M_{BS} = \frac{1}{6}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 0 & 3 & 0\\ 1 & 4 & 1 & 0\end{bmatrix}, G_{BS} = \begin{bmatrix}P_{i-3}\\ P_{i-2}\\ P_{i-1}\\ P_i\end{bmatrix}$$

There are two things to note here.

Firstly we have not made any argument for the matrix itself $M_{BS}$.

Secondly we have a T vector which is a little more complicated than those we know from Hermit and Bezier, since we have $(t-t_i)$ in stead of t. we can, without loosing generality, replace $(t-t_i)$ with t, and thus achieve the same T·M for all curve segments. we know that this product will give us 4 weight functions which in turn tells us how strong influence each of the 4 control points has on the curve when t runs from 0 to 1. For B-Splines the weight functions will be:




$V_{i-3} = 1/6$

$V_{i-2} = 1/6$

$V_{i-1} = 1/6$

$V_i = 1/6$

We use them like this:

$Q_i(t)=$

$V_{i-3} \cdot P_{i-3} +$

$V_{i-2} \cdot P_{i-2} +$

$V_{i-1} \cdot P_{i-1} +$

$V_i \cdot P_i$

This far we have established and described a complete curve form which is uniform in the sense that all curve segments are described in the same way. The t - values are regular in the sense that they "moves us the same amount forward": $t_{i+1} - t_i = 1$

**Objective:**

This paper intends to explore **Nonuniform Rational** B-**Splines** - **NURBS**. B-Splines and Beziers are parallel inventions of the same thing. Where Beziers try to start from the idea of fitting tangents.

**Rational B-splines**

Rational B-splines have all of the properties of non-rational B-splines plus the following two useful features:

- They produce the correct results under projective transformations (while non-rational B-splines only produce the correct results under affine transformations).
- They can be used to represent lines, conics, non-rational B-splines; and, when generalised to patches, can represents planes, quadrics, and tori.

The antonym of *rational* is *non-rational*. Non-rational B-splines are a special case of rational B-splines, just as uniform B-splines are a special case of non-uniform B-splines. Thus, *non-uniform rational B-splines* encompass almost every other possible 3D shape definition. *Non-uniform rational B-spline* is a bit of a mouthful and so it is generally abbreviated to *NURBS*.

We have already learnt all about the the *B-spline* bit of *NURBS* and about the *non-uniform* bit. So now all we need to know is the meaning of the *rational* bit and we will fully(?) understand NURBS.

Rational B-splines are defined simply by applying the B-spline equation (Equation 87) to homogeneous coordinates, rather than normal 3D coordinates. We discussed homogeneous coordinates in the IB course. You will remember that these are 4D coordinates where the transformation from 4D to 3D is:

$$(x', y', z', w) \rightarrow \left(\frac{x'}{w}, \frac{y'}{w}, \frac{z'}{w}\right)$$

Last year we said that the inverse transform was:

$$(x, y, z) \to (x, y, z, 1)$$

This year we are going to be more cunning and say that:

$$(x, y, z) \to (xh, yh, zh, h)$$

Thus our 3D control point, $\mathbf{P}_i = (x_i, y_i, z_i)$, becomes the homogeneous control point, $\mathbf{C}_i = (x_i h_i, y_i h_i, z_i h_i, h_i)$.

A NURBS curve is thus defined as:

$$\mathbf{P}_H(t) = \sum_{i=1}^{n+1} N_{i,k}(t)\mathbf{C}_i, t_{\min} \leq t < t_{\max}$$

Compare Equation 97 with Equation 87 to see just how easy this is!

**NURBS curve**

We now want to see what a NURBS curve looks like in normal 3D coordinates, so we need to apply Equation 94 to Equation 97. In order to better explain what is going on, we first write Equation 97 in terms of its individual components. Equation 97 is equivalent to:

$$x'(t) = \sum_{i=1}^{n+1} x_i h_i N_{i,k}(t)$$

$y'(t) = \sum_{i=1}^{n+1} y_i h_i N_{i,k}(t)$

$z'(t) = \sum_{i=1}^{n+1} z_i h_i N_{i,k}(t)$

$h(t) = \sum_{i=1}^{n+1} h_i N_{i,k}(t)$

Equation tells us that, in 3D:

$x(t) = x'(t)/h(t)$

$y(t) = y'(t)/h(t)$

$z(t) = z'(t)/h(t)$

Thus the 4D to 3D conversion gives us the curve in 3D:

$$\mathbf{P}(t) = \frac{\sum_{i=1}^{n+1} N_{i,k}(t)\mathbf{P}_i h_i}{\sum_{i=1}^{n+1} N_{i,k}(t) h_i}, t_{\min} \le t < t_{\max}$$

This looks a lot more fierce than Equation 97, but is simply the same thing written a different way.

So now, we need to define an additional parameter, $h_i$, for each control point, $\mathbf{P}_i$. The default is to set $h_i = 1, \forall i$. This results in the denominator of Equation 105 becoming one, and the NURBS equation and (1,0), with the other control points at $(\alpha, 1)$ and $(1, \alpha)$. Now see how close this ``quarter circle" comes to the real quarter circle defined by $x^2 + y^2 = 1$, i.e. what is the value of $\alpha$ for which the Bezier curve most closely matches the quarter circle.

**NURBS representation**

NURBS *can* be used to represent circles, and all of the other conics. NURBS surfaces can be used to represent quadric surfaces. As an example, let us consider one way in which NURBS can be used to describe a true circle. Rogers and Adams cover this on pages 371-375.

Construct eight control points in a square. Let $\mathbf{P}_1$, $\mathbf{P}_3$, $\mathbf{P}_5$, and $\mathbf{P}_7$ be the vertices of the square. Let $\mathbf{P}_0$, $\mathbf{P}_2$, $\mathbf{P}_4$, and $\mathbf{P}_6$ be the midpoints of the respective sides, so that the vertices are numbered sequentially as you proceed around the square. Finally, you need a ninth point to join up the curve, so let $\mathbf{P}_8 = \mathbf{P}_0$.

Use a quadratic B-spline basis function with the knot vector

[0,0,0,1,1,2,2,3,3,4,4,4]. This means that the curve will pass through $\mathbf{P}_0$, $\mathbf{P}_2$, $\mathbf{P}_4$, $\mathbf{P}_6$ and $\mathbf{P}_8$, and allows us to essentially treat each quarter of the circle independently.

We finally need to specify the homogeneous co-ordinates. As a circle is symmetrical is should be obvious that that $h_1 = h_3 = h_5 = h_7 = \alpha$ and $h_0 = h_2 = h_4 = h_6 = h_8 = \beta$. As we would like the curve to pass through the even numbered points we know that $\beta = 1$. All we therefore need to determine is $\alpha$, the value of the odd numbered homogeneous co-ordinates.

If $\alpha = 1$ then the NURBS curve will bulge out more than a circle. If $\alpha = 0$, it will bow in. This gives us limits on the value of $\alpha$. To find the exact value we take one quarter of the NURBS curve definition:

$$\mathbf{P}(t) = \frac{(1-t)^2\mathbf{P}_0 + 2\alpha t(1-t)\mathbf{P}_1 + t^2\mathbf{P}_2}{(1-t)^2 + 2\alpha t(1-t) + t^2}, 0 \le t < 1$$

Assume now that $\mathbf{P}_0 = (0,1)$, $\mathbf{P}_1 = (1,1)$, and $\mathbf{P}_2 = (1,0)$. Insert Equation 106 into the equation for the unit circle . The resulting equation is:

$$\frac{((1-t^2+2\alpha t(1-t))^2+(2\alpha t(1-t)+t^2)^2}{((1-t)^2+2\alpha t(1-t)+t^2)^2}=1, 0\leq t<1$$

Now solve this for $\alpha$. Equation 107 is essentially:

$$\frac{a_N t^4+b_N t^3+c_N t^2+d_N t+e_N}{a_D t^4+b_D t^3+c_D t^2+d_D t+e_D}=1, 0\leq t<1$$

From this we can conclude that we require $a_N=a_D$, $b_N=b_D$, $c_N=c_D$, $d_N=d_D$, and $e_N=e_D$. The first three all solve to give the result that $\alpha=1/\sqrt{2}$, while the last two cancel out totally to give the tautology 0=0.

Thus $\alpha=1/\sqrt{2}$. If you are in a hurry to solve this, and you can remember that each of the first three equations gives the same answer then you need simply extract the coefficients of $t^4$ from Equation 107 and solve. It is even easier if you remember the magic number, $1/\sqrt{2}$.

## Conclusion

Non-uniform rational B-splines (NURBS) have become a de facto standard in commercial modeling systems because of their power to represent both free-form shapes and some common analytic shapes. To date, however, NURBS have been viewed as purely geometric primitives, which require the designer to interactively adjust many degrees of freedom (DOFs )-control points and associated weights-to achieve desired shapes. Despite modem interactive devices, this conventional shape modification process can be clumsy and laborious when it comes to designing complex, real-world objects. This thesis paves the way for NURBS to achieve their full potential, by putting the laws of physics on their side. The thesis proposes, develops, and applies dynamic NURBS (D-NURBS), a physics-based generalization of the NURBS representation. D-NURBS unify the features of the industry-standard g~ometry with the many demonstrated conveniences of interaction through physical dynamics.

The mathematical development consists of four related parts: (i) D-NURBS curves, (ii) tensor product D-NURBS surfaces, (iii) swung D-NURBS surfaces, and (iv) triangular DNURBS surfaces. We use Lagrangian mechanics to formulate the equations of motion for all four varieties, and finite element analysis to reduce these equations to efficient algorithms that can be simulated at interactive rates using standard numerical techniques. We implement a prototype modeling environment based on D-NURBS, demonstrating that D-NURBS are effective tools in a wide range of applications in CAD and graphics. We· demonstrate shape blending, scattered data fitting, surface trimming, cross-sectional shape design, shape metamorphosis, and free-form deformation with geometric and physical constraints. Thus, D-NURBS provide a systematic and unified approach for a variety of CAD and graphics modeling problems such as constraint-based optimization, variational parametric design, automatic weight selection, shape approximation, user interaction, etc. They also support direct manipulation and interactive sculpting through the use of forcebased manipulation tools, the specification of geometric constraints, and the adjustment of physical parameters such as mass, damping, and elasticity.

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