Study and some Result on Non expansive Mapping in linear 2 normed spaces.

Dr. Hans Kumar Singh
Department of Physics,
Baboo Bhuneshwar Prasad Degree College, Jai Prakash University, Chapra.

INTRODUCTION — The notion of linear 2- normed spaces was introduced by S. Gahler. He further studies the topological studies of 2-normed spaces. Iseki introduced the notion of non-expansive mapping in 2- normed spaces. Then mathematician like Diminni and white further studied non-expansive mapping in linear 2- normed spaces and obtained the results of Iseki as their corollaries and they contributed a lot for the extension of this branch of mathematics, physics and other Science.

KEYWORD — 2- normed spaces, non-expensive mapping, convex subset

1. Let $X$ be a linear space of dimension greater than 1 and let $\|:\|$ be a real valued function defined on $X \times X$ such that :
   1. $\| a, b \| = 0$ if any only if and $b$ are linearly dependent,
   2. $\| a, b \| = \| b, a \|,$
   3. $\| a, \alpha b \| = |\alpha| \| a, b \|,$ were $\alpha$ is real,
   4. $\| a + b, c \| \leq \| a, b \| + \| a, c \|.$

   $\|:\|$ is called a 2-norm on $X$ and $(X, \|:\|)$ is a linear 2-normed space. By condition 2 and 4, a 2-norm is non-negative.

Definition : If $K$ is a convex subset of $X$, a mappings $T : K \rightarrow X$ is said to be non-expansive if for every $x, y \in K$ and $z \in X$,

1. $\| T(x) - T(y), z \| \leq \| x - y, z \|.$

In the following, the real number system will be denoted by $R$. Also, a subset of $L$ of $x$ of the form $\{x_1 + ax_2 : a \in R\}$, where $x_2$ is non-zero, will be called a line. $\alpha \in R$

Theorem : Let $K$ be a convex set which contains a least 2 elements and is none a subset of line. Then, $T$ is non-expansive if and only if there is a $c \in R$ and there is a point $z_0 \in X$ such that $|c| < 1$ and $T(x) = cx + z_0$, for every $x \in K$.

Proof— Since all functions of the above type are non-expansive, we need show only that all non-expansive maps are of this type.

1. Assume first the $0 \in K$ and $T(0) = 0$. Then, for every $x \in X$,
2. $\| T(x), z \| \leq \| X, Z \|.$

Therefore, for each $x \in K$, there is a real number $g(x)$ such that $T(x) = g(x)x.$
If \( x \) and \( y \) are independent elements of \( K \), then \( \frac{1}{2} (x + y) \in K \) also, and by (1),
\[
\left( T \frac{x + y}{2} \right) - T(x), x - y \| \leq \| \frac{x + y}{2} x - y \| = 0.
\]
Therefore, there is a \( k \in R \) such that
\[
\left( T \frac{x + y}{2} \right) - T(x) = k(x - y)
\]
\[
g \left( \frac{x + y}{2} \right)\left( \frac{x + y}{2} \right) - g(x) x = k(x - y)
\]
Then,
\[
\left[ \frac{1}{2} g \left( \frac{x + y}{2} \right) - g(x) - k \right] x = \left[ k + \frac{1}{2} g \left( \frac{x + y}{2} \right) \right] y
\]
which implies that \( g(x) = g \left( \frac{x + y}{2} \right) \) by the independence of \( x \) and \( y \). Since a similar argument shows that \( g(y) = g \left( \frac{x + y}{2} \right) \), it follows \( g(x) = g(y) \) whenever \( x \) and \( y \) are independent.

If \( x \) and \( y \) are non-zero, independent elements of \( K \), then since \( K \) is not a subset of a line, there is a \( z \in K \) such that \( z \) and \( x \) and \( z \) and \( y \) are independent. By the arguments used above, \( g(x) = g(z) = g(y) \).

Therefore, \( g(x) = g(y) \) for all non-zero \( x, y \in K \). Since \( T(0) = 0 \), there is a real number \( c \) such that \( T(x) = cx \) for every \( x \in K \). Finally, (2) implies that \( |c| < 1 \).

2. For arbitrary \( T \) and \( K \) which satisfy the hypotheses, choose and \( x \in K' = \{ x - x_0 : x \in K \} \).

Then \( K' \) is not contained in a since \( K \) is not a subset of a line, and \( x \in K' \). Define \( S : K' \to x \) by
\[
\| S(x - x_0) - S(y - x_0), z \| = \| T(x) - T(y), z \|< \|x - y, z\|
\]
\[
= \|(x - x_0) - (y - x_0), z\|.
\]
Hence, \( S \) is non-expansive on \( K' \) and
\[
S(0) = S(x - x_0) = T(x_0) - T(x_0) = 0
\]
By part 1, there is a \( c \in R \) such that \( |C| < 1 \) and for every \( x \in K \),
\[
S(x - x_0) = c (x - x_0).
\]
Therefore, for every \( x \in K \),
\[
T(x) = cx + T(x_0) - x_0.
\]
The following example shows that the characterization fails if \( K \) is contained in a line.

**Example:** Suppose \( K \) is subset of the line \( L = T(x) = cx + T(x_0) - x_0 \).

Define \( T : K \to X \) by \( T(x_1 + \alpha x_2) = (\sin \alpha) x_2 \).
Then, if \( x_1 + \alpha x_2 \) and \( x_1 + \gamma x_2 \) are in \( K \) and \( z \in X \),
\[
\| T(x_1 + \alpha x_2) - T(x + \gamma x_2), z \| = \| \sin \alpha - \sin \gamma \| ||x_2, z|| < \alpha - \gamma \| ||x_2, z||
\]

Hence, \( T \) is a non-expansive mapping which does not satisfy Theorem 1.

For convex sets which are subsets of lines, we have the following characterization of non-expansive mappings.

**Theorem:** Suppose \( K \) is a convex subset of line \( L = \{ x_1 + \alpha x_2 : \alpha \in R \} \), where \( x_1 \in K \), and let \( \{ \alpha : x_1 + \alpha x_2 \in R \} \). Then, \( T : K \rightarrow X \) is non-expansive if and only if there is a function \( g : A \) \( g(0) = 0 \) and \( T(x_1 + \alpha x_2) = g(\alpha) x_2 + T(x_1) \).

**Proof:** Again, since the sufficiency of the above conditions is clear, we need only to prove the necessity.

1. Assume \( x_1 = 0 \) and \( T(0) = 0 \). Then, for every \( \alpha \in A \) and \( z \in X \), (3) \( ||T(x_2), z|| < ||\alpha x_2, z|| \).

   Therefore, for every non-zero \( \alpha \in A \), there is a real number \( g(\alpha) \) such that \( |g(\alpha) - g(\gamma)| < \alpha - \gamma| \) for every \( \alpha, \gamma \in A \).

2. If \( x_1 = 0 \) or \( T(x_1) = 0 \) let \( K' = \{ \alpha x_2 : \alpha \in A \} \). Then, \( K' \) is convex, \( 0 \in K' \), and \( K' = \{ \alpha x_2 : \alpha \in R \} \). Define \( S : K' \rightarrow X \) by
   \[
   S(\alpha x_2) = T(x_1 + \alpha x_1) - T(x_1)
   \]
   for every \( \alpha \in A \). Note that \( S(0) = 0 \) and for \( \alpha, \gamma \in A \) and \( z \in X \),
   \[
   ||S(\alpha x_2) - S(\gamma x_2), z|| = ||T(x_1 + \alpha x_2) - T(x_1 + \gamma x_2), z|| < ||\alpha x_2 - \gamma x_2, x||.
   \]

   Therefore, since \( S \) and \( K' \) satisfy the assumptions made in part 1, it follows that there is a function \( g : A \rightarrow R \) such \( S(\alpha x_2) = g(\alpha) x_2 \). Hence, for every \( \alpha \in A \), \( T(x_1 + \alpha x_2) = g(\alpha) x_2 + T(x_1) \).

   It is known that in a strictly convex 2-normed space, the set \( F(T) \) of fixed points of a non-expansive \( T \) is always a convex set. This result can now be proven for any 2-normed space.

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