Min-Min Operation on Intuitionistic Fuzzy Matrix

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Abstract: In this paper Min-Min operation on IFMs and study conditions for convergence powers of transitive IFM are introduced.

Keywords and Phrases: Intuitionistic fuzzy set (IFS), Intuitionistic fuzzy implication operator (IFIO), Intuitionistic fuzzy matrix (IFM)

I. INTRODUCTION

Since Zadeh [11] introduction of fuzzy sets, Atanassov [1] generalized the concept of fuzzy sets into intuitionistic fuzzy set (IFS) in X (universal set) is defined as an object of the following from \( A = \{(x, \mu_A(x), \gamma_A(x))/x \in X \} \) where the functions: \( \mu_A(x): x \rightarrow [0,1] \) and \( \gamma_A(x): x \rightarrow [0,1] \) define the membership function and non-membership function of the element \( x \in X \) respectively and for every \( x \in X: 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \). Xu, Yager [10] defined an Intuitionistic Fuzzy Matrix (IFM). \( AsA = \{a_{ij} \} \) where \( a_{ij} \) and \( a_{ij}^\prime \) denote the membership and non-membership value respectively.

After the introduction of Fuzzy Matrix (FM) theory using Max-Min algebra by Thomson [9], Bhowmik and Pal [3] studies the convergence of the Max-Min of an IFM by Hashimoto [4] and several others have studied the convergence of power of a fuzzy transitive matrix. Further, the Max-Min operation has been extended to IFM. Atanassov [2] used implication operators in IFSs. Sriram and Murugadas [8] used implication operator for IFM and studied concept of \( g^{-1} \)-inverse and semi-inverse of an IFM which was a generalization of FM studied. Murugadas and Lalitha [5] used hook implication operator for IFMs as well as for IFM. Muthuraj, Sriram and Murugadas [6] used min-min composition of IFM. Riyaz Ahmad paddar and Murugadas [7] Max-Max operation on Intuitionistic fuzzy Matrix.

In this paper we introduce Min-Min operation directly to IFMs which is more relevant than Max-Min operation. For example, consider two IFMs \( A \) and \( B \) such that

\[
A = \{(3,2), (4,1)\} \quad \text{and} \quad B = \{(0,3,0,4), (0,5,0,2)\}
\]

Then \( \text{Max-Min} AB = \{(0,3,0,4), (0,3,0,2)\} \)

Then \( \text{Max-Min} AB = \{(0,2,0,5), (0,3,0,6)\} \)

Thus \( \text{Min-Min} AB \leq \text{Max-Min} AB \).

II. PRELIMINARIES

Let \( (x,x'), (y,y') \in IFS \) then \( (x,x') \lor (y,y') = (\text{Min}(x,y), \text{Max}(x',y')) \)

For any two comparable elements \( (x,x'), (y,y') \in \text{IFS} \) the operation \( (x,x') \rightarrow (y,y') \) is defined as

\[
(x,x') \rightarrow (y,y') = \begin{cases} (1,0) & \text{if } (x,x') \geq (y,y') \\ (x',y') & \text{if } (x,x') < (y,y') \end{cases}
\]

For \( n \times n \) intuitionistic fuzzy matrices \( A = \{a_{ij},a_{ij}^\prime\} \) and \( P = \{p_{ij},p_{ij}^\prime\} \)

\[
A \land P = \{(a_{ij} \land p_{ij}, a_{ij}^\prime \lor p_{ij}^\prime)\}
\]

\[
A \lor P = \{(a_{ij} \lor p_{ij}, a_{ij}^\prime \land p_{ij}^\prime)\}
\]

Here \( A \lor P, A \land P \) are equivalent to \( A + P, A \cap P \) the component wise addition and component wise multiplication \( A \land P \) respectively.

\[
A \times P = (a_{ij} a_{ij}^\prime \land p_{ij}, p_{ij}^\prime) \lor (a_{ij} a_{ij}^\prime \lor p_{ij}, p_{ij}^\prime) \lor \cdots \lor (a_{in} a_{in}^\prime \lor p_{nj}, p_{nj}^\prime)
\]

Here \( \rightarrow \) represents component wise comparison of \( A, P \) using \( \rightarrow \).

\[
A^k = \{(\delta_{ij}, \delta_{ij}^\prime)\} \text{ where } (\delta_{ij}, \delta_{ij}^\prime) = (1,0) \text{ if } i = j \text{ and } (\delta_{ij}, \delta_{ij}^\prime) = (0,1) \text{ if } i \neq j.
\]

\[
A^k = A^k \times A, k = 0,1,2,\ldots
\]

\[
A \leq P(P \geq A) \text{ if and only if } (a_{ij}, a_{ij}^\prime) \leq (p_{ij}, p_{ij}^\prime) \text{ for all } i,j.
\]

If \( A \geq I_n \), then \( A \) is reflexive IFM where in the \( n \times n \) identity IFM. \( A = \{a_{ij}, a_{ij}^\prime\} \) is weakly reflexive IFM if and only if \( (a_{ij}, a_{ij}^\prime) \geq (a_{ij}, a_{ij}^\prime) \) for all \( i,j = 1,2,\ldots,n \).

Throughout we deal with intuitionistic fuzzy matrices. A matrix \( A \) is transitive if \( A^2 \leq A \). This matrix represents a intuitionistic fuzzy transitive relation. The above definition of transitivity is equivalent to what is called Max-Min transitivity. That is, matrix \( A = \{a_{ij}, a_{ij}^\prime\} \) is transitive if and only if \( \text{Min}(a_{ik},a_{ik}^\prime), (a_{kj},a_{kj}^\prime) \leq (a_{ij}, a_{ij}^\prime) \) for all \( k \). This definition is most basic and seems to be convenient when intuitionistic fuzzy matrices are generalized to certain matrices over other algebras.

III. SOME RESULTS

I define Min-Min operation on IFM and exhibit some interesting results. In the following, let \( A = \{a_{ij}, a_{ij}^\prime\} \) and \( P = \{p_{ij}, p_{ij}^\prime\} \) be IFM of order \( n \times n \) and the entries in \( A \) and \( P \) are comparable.

Definition 3.1 For IFMs \( A \) and \( P \), define the Min-Min product of \( A \) and \( P \) as

\[
A \cdot P = \left( \bigwedge_{k=1}^{n} (a_{ik} \land p_{kj}) \lor \bigvee_{k=1}^{n} (a_{ik}^\prime \lor p_{kj}) \right)
\]
Let $A \ast P$ denote the Min-Min product of the IFMs $A$ and $P$.

Clearly $A \ast P$ is also an IFM, $\ast$ is associative and $\ast$ is distributive over addition ($\ast$). Also the set of all IFM under $\ast$ and $\ast$ from a semi-ring.

**Theorem 3.2** If $A$ is an $\times n$ transitive matrix, then $(A \ast (A \times P))^n = (A \ast (A \times P))^{n+1}$ for any $n \times n$ IFMP.

**Proof.** Let $S = (s_{ij}, s_{ij}^\prime) = A \ast (A \times P)$, that is

$$s_{ij} = \left(\frac{1}{n} \sum_{k=1}^{n} (a_{ik} \wedge p_{kj}) \wedge (a_{ik} \vee p_{kj})\right).$$

1. Assume that there exist indices $l_1, l_2, \ldots, l_{n-1}$ such that

$$(s_{ij}, s_{ij}^\prime) \wedge (s_{ij}^\prime, s_{ij}^\prime) \wedge \cdots \wedge (s_{in-1, j}, s_{in-1, j}^\prime) = (f, f^\prime) < (1,0).$$

Let $l_o = i$ and $l_{n-1} = j$. Then $l_k = l_b$ for some $a > b$. We define $(h, h^\prime)$ by

$$(h, h^\prime) = (a_{ia_{ib'}, a_{ia_b'}}, a_{ia_{ib'}, a_{ia_b'}}) \wedge \cdots \wedge (a_{ib_{a_b'}, a_{ib_{a_b'}}} \wedge a_{ia_{ib'}, a_{ia_b'}})$

where $a \geq b > 0$.

Then $(h, h^\prime) = (a_{im-1, m}, a_{im-1, m}^\prime) < (\frac{1}{n} \sum_{k=1}^{n} (a_{imk} \wedge p_{kim}), \frac{1}{n} (a_{imk} \vee p_{kim})).$

If $(a_{im, l_1}, s_{im}^\prime) \geq (\frac{1}{n} \sum_{k=1}^{n} (a_{imk} \wedge p_{kim})), (a_{imk} \vee p_{kim})$, then $(h, h^\prime)$ is $\geq (a_{imk}, a_{imk}') \wedge (a_{im}, a_{im}^\prime) \wedge \cdots \wedge (a_{im}, a_{im}^\prime) = (h, h^\prime)$ for some $k_1$. Since $(a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) = (h, h^\prime)$ we have

$$(a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) = (a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) \wedge (a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) = (h, h^\prime)$$

Thus,

$$(\frac{1}{n} \sum_{k=1}^{n} (a_{imk} \wedge p_{kim})), (\frac{1}{n} (a_{imk} \vee p_{kim}))) \leq (a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) \wedge (p_{kim}, p_{kim}^\prime) = (h, h^\prime)$$

which is contradiction. So,

$$(a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) < (\frac{1}{n} \sum_{k=1}^{n} (a_{imk} \wedge p_{kim})), (\frac{1}{n} (a_{imk} \vee p_{kim}))) = (h, h^\prime).$$

Hence $(s_{im}^\prime, s_{im}^\prime) \leq (h, h^\prime) \leq (g, g^\prime).$

Therefore $(s_{im}^\prime, s_{im}^\prime) \leq (g, g^\prime).$

2. Assume that there exist indices $l_1, l_2, \ldots, l_n$ such that

$$(s_{ij}^\prime, s_{ij}^\prime) \wedge (s_{ij}^\prime, s_{ij}^\prime) \wedge \cdots \wedge (s_{in'}^\prime, s_{in'}^\prime) = (g, g^\prime) < (1,0).$$

Let $l_0 = i$ and $l_{n+1} = j$.

(a) Assume $l_a = l_b = l_c$ where $a > b > c$. Then we have

$$(s_{im}^\prime, s_{im}^\prime) \leq (g, g^\prime), a > m \geq b$$

Thus,

$$s_{im}^\prime = \left(\frac{1}{n} \sum_{k=1}^{n} (a_{imk} \wedge p_{kim}), (\frac{1}{n} (a_{imk} \vee p_{kim})\right) \leq (a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) \wedge (p_{kim}, p_{kim}^\prime) = (h, h^\prime)$$

(b) Assume $l_a = l_b = l_c = l_d$

(i) If $a > b > c > d$ then $(s_{im}^\prime, s_{im}^\prime) \leq (g, g^\prime), a > m \geq b$ for some $l_m$.

Thus,

$$(s_{im}^\prime, s_{im}^\prime) \wedge (s_{im}^\prime, s_{im}^\prime) \geq (g, g^\prime), a > m \geq b$$

Therefore $(s_{im}^\prime, s_{im}^\prime) \leq (g, g^\prime).$

(ii) If $a > b > d$ then $(s_{im}^\prime, s_{im}^\prime) \leq (g, g^\prime), a > m \geq b$ for some $l_m$

where

$$(h, h^\prime) = (a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) = (a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) \wedge (a_{ib_{a_b'}, a_{ib_{a_b'}}}^\prime) \wedge (a_{ib_{a_b'}, a_{ib_{a_b'}}}^\prime)$$

$$(h, h^\prime) = (a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) \wedge (a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) = (h, h^\prime)$$

Thus,

$$(a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) \leq (a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) \wedge (p_{kim}, p_{kim}^\prime) = (h, h^\prime)$$

which contradicts the fact that

$$(h, h^\prime) = (a_{im, l_1 \ast m}, a_{im, l_1 \ast m}^\prime) < 0$$

So $(s_{im}^\prime, s_{im}^\prime) \leq (g, g^\prime).$

Hence $(s_{im}^\prime, s_{im}^\prime) \leq (g, g^\prime), a > m \geq b$ for some $l_m$.

It is clear that $(s_{ij}^\prime, s_{ij}^\prime) \leq (g, g^\prime)$ for $m > c$ (or $d$) $\geq m$. Suppose that $c \geq m \geq d$.

By the same argument as in (ii) we have

$(s_{im}^\prime, s_{im}^\prime) \leq (g, g^\prime)$ then
\[(s_{lm_a}(a), s_{lm_a}(a')) V (s_{lm_m}, s_{lm_m}) V (s_{lm_m}(m-a), s_{lm_m}(m-a)) V (s_{lm_m}((m+n-1)-m), s_{lm_m}((m+n-1)-m)) \leq (g, g').\]

\[\square\]

**Example 3.3** \[A = \begin{pmatrix} (1,0) & (0,1,0.5) \\ (0,3,0,5) & (1,0) \end{pmatrix}\] and 
\[P = \begin{pmatrix} (0,1,0.5) & (0,5,0.2) \\ (0,4,0,5) & (0,3,0,2) \end{pmatrix}\]
\[A \times P = \begin{pmatrix} (1,0) & (0,1,0,5,5) \\ (0,3,0,5) & (1,0) \end{pmatrix}\]
\[A \times P = \begin{pmatrix} (0,1,0,5) & (0,5,0,2) \\ (0,4,0,5) & (0,3,0,2) \end{pmatrix}\]
\[A^2 = A \cdot A = \begin{pmatrix} (1,0) & (0,1,0,5) \\ (0,3,0,5) & (1,0) \end{pmatrix}\]
\[S = \begin{pmatrix} (1,0) & (0,1,0,5) \\ (0,3,0,5) & (1,0) \end{pmatrix}\]
\[S = \begin{pmatrix} (1,0) & (0,1,0,5) \\ (0,3,0,5) & (1,0) \end{pmatrix}\]

Then \[S^3 = S^2 \times S = \begin{pmatrix} (1,0) & (0,1,0,5) \\ (0,3,0,5) & (1,0) \end{pmatrix}\]

Thus we have \[S^3 = S^2\]

From Theorem 3.2, we get the following two results.

**Corollary 3.4** If \(A\) is ann \(\times \) ntransitive matrix, then \(A \prec (P \times A)^n = (A \prec (P \times A))^{n+1}\) for any \(n \times n\) matrix \(P\).

**Corollary 3.5** Let \(A\) be ann \(\times \) ntransitive matrix, then \(A^n = A^{n+1}\).

We now consider conditions under which our \(n \times n\) transitive matrix \(A\) fulfills the relationship \(A^{n-1} = A\), where \(n \geq 2\).

**Theorem 3.6** Let \(A\) be an \(n\times n\) transitive matrix and \(I \geq P \geq A\) and the Min-Min Product \(A \cdot A^T \leq ((a_{ij}, a_{ij}'))\) for some \(P^{-1} = P^n\).

**Proof.** First we know that \(P^{-1} = P^n\) suppose that \((p_{ij}(n-1), p_{ij}(n-1)) = (c, c') \leq (1,0)\)

Then there exists \(k_1, k_2, \ldots, k_{n-2}\) such that \(P_{i,k_1, k_1} P_{k_1, k_2} \cdots P_{k_{n-2}, k_{n-2}} = (c, c')\)

Thus \((a_{i,k_1, k_1}, a_{k_1, k_2}, a_{k_1, k_2}) \cdots a_{k_{n-2}, k_{n-2}} = (c, c')\)

Let \(k_0 = i + k_{n-1} = j\)

(a) If \(k_0 = k_0\) for some \(a\) and \(b(a > b)\), then \(p_{k_0,k_0}(a-b), p_{k_0,k_0}(a-b) = (c, c')\)

Thus \((a_{i,k_1, k_1} a_{k_1, k_2}(a-b)) \leq (c, c'), (a_{k_1, k_2} a_{k_1, k_2}(a-b)) \leq (c, c')\)

So \((p_{i,k_1, k_1}) V (p_{k_1, k_2}) V (p_{k_1, k_2}) \cdots p_{k_{n-2}, k_{n-2}} = (c, c')\)

Hence \((p_{ij}(n), p_{ij}(n)) \leq (c, c')\)

(b) Suppose that \(k_0 = k_0\) for all \(a \neq b\). By hypothesis,

\[
\frac{n}{k_1 = 1} \left( A_{i,k_1} A_{k_1,m} a_{k_1,m} \right) \geq a_{k_0,m} a_{k_0,m} \text{ for some } m.
\]

Then \((a_{k_0,m} a_{k_0,m} a_{k_0,m}) \leq (c, c')\)

Thus \((p_{i,k_1, k_1}) V (p_{k_1, k_2}) V (p_{k_1, k_2}) \cdots p_{k_{n-2}, k_{n-2}} = (c, c')\)

So \((p_{ij}(n), p_{ij}(n)) \leq (c, c')\)

(2) Next we show that \(P^n P^{n-1}\)

Let \((p_{ij}(n), p_{ij}(n)) = (c, c') \leq (1,0)\)

Then there exists \(k_1, k_2, \ldots, k_{n-1}\) such that \(p_{i,k_1, k_1} V (p_{k_1, k_2}) V (p_{k_1, k_2}) \cdots p_{k_{n-1}, k_{n-1}} = (c, c')\)

Let \(k_0 = i + k_{n-1} = j\). Then \(a_{k_0,k_0}(a-b), a_{k_0,k_0}(a-b) = (c, c')\)

Hence \((p_{i,k_1, k_1}) V (p_{k_1, k_2}) V (p_{k_1, k_2}) \cdots p_{k_{n-1}, k_{n-1}} = (c, c')\)

\[
\frac{(p_{i,k_1, k_1}) V (p_{k_1, k_2}) V (p_{k_1, k_2}) \cdots p_{k_{n-1}, k_{n-1}}) \leq (c, c')
\]
Therefore \((p_{ij}^n - 1), p_{ij}^{n+1} - 1) \leq (c, c')\).

**Example 3.7** \(A = \begin{pmatrix} 0.1,0.2 & 0.0,3 \\ 0.3,0.2 & 0.0,3 \end{pmatrix}, P = \begin{pmatrix} 0.3,0.2 & 0.0,2 \\ 0.4,0.1 & 0.1,0,2 \end{pmatrix}\)

\(A^2 = \begin{pmatrix} 0.1,0.2 & 0.0,3 \\ 0.3,0.2 & 0.0,3 \end{pmatrix} \leq A\). (A is transitive)

\(A^3 = \begin{pmatrix} 0.1,0.2 & 0.0,3 \\ 0.1,0,2 & 0.0,3 \end{pmatrix} = A^2\)

\(p^2 = \begin{pmatrix} 0.3,0,2 & 0.0,2 \\ 0.4,0,1 & 0.1,0,2 \end{pmatrix}\)

\(p^2p = \begin{pmatrix} 0.3,0,2 & 0.0,2 \\ 0.3,0,2 & 0.1,0,2 \end{pmatrix}\)

\(p^3 = \begin{pmatrix} 0.3,0,2 & 0.0,2 \\ 0.3,0,2 & 0.1,0,2 \end{pmatrix}\)

\(p^3 = p^2\)

**Theorem 3.8** If \(A\) is an \(n \times n\) transitive matrix, \(A \land I \geq P \geq A\) and \(p \cdot p^T \leq (p_{ij}, p_{ij}')\) for some \(j\), then \(p^{n-1} = p^n\).

As a special case of Theorem 3.6 or Theorem 3.8 we obtain the following corollary where \(A\) is a transitive IFM.

**Corollary 3.9** If \(A\) is an \(n \times n\) transitive intuitionistic fuzzy matrix.

\(A \cdot A^T = (a_{ij}, a_{ij}')\) for some, then \(A^{n-1} = A^n\).

**REFERENCES**


[8] Sriram and Murugadas P. 2011, Contribution to a study on Generalized Fuzzy Matrices, Ph.D. Thesis, Department of Mathematics, Annamalai University, Tamil Nadu, India.

