OPTIMAL SOLUTION OF AN ASSIGNMENT PROBLEM AS A SPECIAL CASE OF TRANSPORTATION PROBLEM

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ABSTRACT: Assignment assume a vital part when relegating employments to the specialists. It is exceptionally vital target in mathematics and is additionally discussed in genuine physical world. In this paper, we examine another method for solving an assignment problem. Additionally, the numerical illustrations has been given to comprehend the procedure of proposed algorithm.

KEY WORDS: Assignment problem, resources, jobs, L-shape of cost matrix consisting of pivot elements in the equivalent reduced row echelon form of the system, optimality.

INTRODUCTION:

The Assignment problem is an exceptionally uncommon instance of Transportation problem for linear programming problem, in which number of offices are to be doled out to an equivalent number of employments, where every administrator can do just a single operation at any given moment. There are such a large number of down to earth circumstances in which issue moves toward becoming to dole out or allot every asset to just a single action (occupation) and the other way around with the end goal that the powerful estimation has been figured.

The Assignment problem produces when the assets those are accessible, for example, men, and machines and so on., have fluctuating level of productivity for performing distinctive exercises. That is the reason the aggregate cost, benefit or time of performing distinctive exercises is additionally not equivalent. In this way the issue turns out to be much logical that, how the assignments ought to be made such that it will optimize the given objective. In reasonable circumstances, where the assignment problem might be valuable: 1) specialists to machines 2) assistants to different checkout counters 3) business people to various deals zones 4) classes to different checkout counters 5) Vehicles to highways 6) contracts to bidders.

Koing (1931) has created and presented a strategy for solving Assignment problem. He gave his name as Hungarian technique. His fundamental expect to find the optimal solution of an Assignment Problem without making an immediate correlation of each arrangement. His strategy takes a shot at the essential of diminishing the given cost grid to a framework of chance cost. The Hungarian technique portrays the calculation, gives ideal arrangement from a limited arrangement of arrangements which takes care of the task issue in polynomial time and which foreseen later primal-double strategies.

James Munkers (1957) surveyed the calculation and watched that it is (unequivocally) polynomial. From that point forward the calculation has been referred to likewise as Kuhn-Munkers calculation or Munkers assignment algorithm.

Thompson (1981) presented a Repetitive method for solving assignment problem which is a bounded non-simplex method for tackling Assignment Problems. It has been demonstrated that the line duals are non-expanding and the segment duals non-diminishing.

Li ET. al. (1997) built up another calculation for the Assignment Problem which additionally called another option to the Hungarian Method. So far in the literature, there are four techniques as: Enumeration method, Simplex method, Transportation method and Hungarian technique for illuminating Assignment Problem.

THEORETICAL DEVELOPMENT:

The general information grid for an Assignment Problem is appeared in the accompanying table. It might be noticed that this data matrix is the same as the Transportation cost matrix with the exception of that the supply (or accessibility) of each of the assets and the request at each of the goals is taken to be one. It is because of the way that assignments are made on a one-to-one basis.

Resources (workers)	Activities (jobs)				Supply
	J_1	J_2	....	J_n	
w_1	C_{11}	C_{12}	....	C_{1n}	1
w_2	C_{21}	C_{22}	....	C_{2n}	1
.	.	.	....		.
.	.	.	....	.	.
.	.	.	....	.	.
w_n	C_{n1}	C_{n2}	....	C_{nn}	.
Demand	1	1	....	1	n

Suppose, x_{ij} represents the assignment of resource (facility) i to the activity (job) j such that

$$x_{ij} = \begin{cases} 1, if\ resource\ i\ is\ assigned\ to\ activity\ j \\ 0\ , otherwise \end{cases}$$

Then the mathematical model of the assignment problem can be stated as:
Minimize $Z=\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}x_{ij}$
Subject to the Constraints: $\sum_{j=1}^{n} x_{ij} = 1$, for all i (resource availability)

$\sum_{i=1}^{n} x_{ij} = 1$, for all j (activity requirement)
And $x_{ij} = 0 \; or \; 1$, for all i and j, where C_{ij} represents the cost of assignment of resource i to activity j.

This Mathematical model of an Assignment problem is a specific instance of the transportation problem for two reasons: 1) the cost matrix is a square lattice and 2) the optimal solution matrix for the problem would have just a single Assignment in a given line (row) or a segment (column).

First, let us convert the constraints of the transportation problem into our standard matrix form for a linear programming problem, $AX = B$

$$X=[x_{11},\ldots\ldots,x_{1n,},x_{21}\ldots\ldots,x_{2n},\ldots\ldots,x_{mn}]$$

$$B=[1,\ldots\ldots,1,1\ldots\ldots,1] \qquad (1)$$

If the Constraints are written as,

$$\begin{aligned}
&x_{11}+x_{12}+\ldots\ldots\ldots+x_{1n} && =1_1,\\
&x_{21}+\ldots\ldots+x_{2n} && =1,\\
&\ddots\ \vdots\\
&x_{m1}+\ldots\ldots+x_{mn} && =1,\\
&x_{11} \qquad +x_{21}+\ldots\ldots\ldots+ \qquad x_{m1} && =1,\\
&\ddots\\
&\ddots\\
&x_{1n} \quad + \quad x_{2n}+ \quad \ldots\ldots\ldots\ldots\ldots\ldots+x_{mn} && =1, \qquad (2)
\end{aligned}$$

Now the equivalent reduced row echelon form of the above system – (2) is as follows,

$$\begin{aligned}
&x_{11}+x_{12}+.x_{13}+ \quad \ldots\ldots\ldots+x_{1n} =1,\\
&x_{12}+\ldots\ldots\ldots\ldots+x_{22}+\ldots\ldots\ldots\ldots\ldots+x_{32}+\ldots\ldots\ldots\ldots\ldots+x_{m2}=1\\
&x_{13}+\ldots\ldots\ldots\ldots+x_{23}+\ldots\ldots\ldots\ldots\ldots+..x_{33}+\ldots\ldots\ldots+x_{m3}=1\\
&x_{14}+\ldots\ldots\ldots+x_{24}+\ldots\ldots\ldots\ldots+x_{34}+\ldots\ldots\ldots\ldots+x_{m4}=1\\
&\ddots\ldots\ldots\ldots\ldots\ddots\ldots\ldots\ldots\ldots\ldots\ddots\ldots\ldots\ldots\ldots\ldots\ddots\ldots.=\vdots\\
&x_{1n}+\ldots\ldots\ldots+x_{2n}+\ldots\ldots\ldots+x_{3n}+\ldots\ldots+x_{mn}=1\\
\\
&x_{21}+.x_{22}+x_{23}\ldots\ldots+x_{2n}=1,\\
&\ddots\ \vdots\\
&x_{m1}+x_{m2}+x_{m3}+\ldots\ldots\ldots+x_{mn}=1 \qquad (3)
\end{aligned}$$

ALGORITHM:

Step: 1 Construct the cost matrix from the given problem, if the number of rows are not equal to the number of columns, then add required number of dummy rows or columns. The cost element in dummy rows/columns are always zero.

Step: 2 1) Identify the smallest element in the **first row only** and subtract it from each element of the **first row only**.

2) Identify the smallest element in the **first column only** and subtract it from each element of the **first column only**.

Step: 3 1) Now identify the minimum value in each row of cost matrix and then subtract it from each element of that row **except first row and first column (i.e. 'L-shape' of the cost matrix).**

2) In the reduced matrix which is gained from 3(1), identify the smallest element in each column and subtract it from each element of that column, **except first row and first column(i.e. 'L-shape' of the cost matrix).** In this step each row and column of whole cost matrix have at least one zero element.

Step: 4 The procedure of making assignments is as follows:

a) **I)** Identify rows successively from top to bottom until a row with exactly one zero element is found. Make an assignment to this single zero by making a square (**☐**) around it. Then cross off (×) all other zeros in the corresponding column.

II) Identify columns successively from left to right hand with exactly one zero element that has not been assigned. Make an assignment to this single zero by making a square (**☐**) around it. Then cross off (×) all other zeros in the corresponding row.

b) **I)** if a row and/or column has two or more unmarked zeros and one can't be chosen by inspection then choose zero element arbitrarily for assignment.

II) Repeat steps (a) and (b) until one of the following optimality arise.

Step: 5 Test of Optimality:

- **a)** If all zero elements in the cost matrix are either marked with square (☐) or are crossed off (×) and there is exactly one assignment in each row and column, then it is an optimal solution. The total cost associated with this solution is obtained by adding the original cost elements in the occupied cells.
- **b)** If a zero element in a row or column was chosen arbitrarily for assignment in step 5(a), there exists an alternative optimal solution.
- **c)** If there is no assignment in a row (or column), then this implies that the total number of assignments are less than the number of rows/columns in the square matrix. In such type of situation follow the step:6

Step: 6 Revise the Optimality cost matrix:

Make a set of horizontal and vertical lines to cover all the zeros in the revised cost matrix obtained from step-4, by using the following procedure:

1) Identify the rows and columns and mark a tick (✓) in which the maximum zero elements (crossed off (×) or allocated cells) such that total number of rows and columns must be equal to the number of assignment has been made as per step-5(c).
2) Draw a straight line through each marked column and each marked row.

Step: 7 Develop the new revised opportunity cost matrix.

1) Among the elements in the matrix not covered by any line, choose the minimum element. Define this value 't'
2) Subtract t from every element in the matrix that is not covered by a line.
3) Add 't'to every element in the matrix covered by the two lines that is intersection of two lines.
4) Elements in the matrix covered by one line and '0' remain unchanged.

Step: 8 Repeat steps 4 to 7 until an optimal solution is obtained.

NUMERICAL ILLUSTRATIONS:

1) A Solicitors' firm employs typists on hourly piece-rate basis for their daily work. There are five typists and their charges and speed are different. According to an earlier understanding only one job was given to one typist and the typist was paid for a full hour, even if he worked for a fraction of an hour. Find the least cost allocation for the following data:

Typist	Rate per hour(Rs.)	No.of pages Typed/Hour
A	5	12
B	6	14
C	3	8
D	4	10
E	4	11

Job	No. of Pages
P	199
Q	175
R	145
S	298
T	178

Step: 1 Construct the cost matrix from the given problem

	P	**Q**	**R**	**S**	**T**
A	85	75	65	125	75
B	90	78	66	132	78
C	75	66	57	114	69
D	80	72	60	120	72
E	76	64	56	112	68

Here the number of rows are equal to number of columns i.e. the problem is Balanced Assignment problem.

Step: 2 1) Identify the minimum element the first row and **first row only** and subtract it from each element of that **first row only**.

20	10	0	60	10
90	78	66	132	78
75	66	57	114	69
80	72	60	120	72
76	64	56	112	68

3) Identify the minimum element in the **first column only except from first row** and then subtract it from each element of that **first column only**.

20	10	0	60	10
15	78	66	132	78
0	66	57	114	69
5	72	60	120	72
1	64	56	112	68

Step: 3 1) Now Identify the smallest element in each row of cost matrix and subtract it from each element of that row **except first row and first column (**i.e. **'L'- shape** of cost matrix).

20	10	0	60	10
15	12	0	66	12
0	9	0	57	12
5	12	0	60	12
1	8	0	56	12

2) In the reduced matrix which is obtained from step-3(1), identify the smallest element in each column and then subtract it from each element of that column, expect first row and first column (i.e.**’- shape** of cost matrix). In this step each row and column of whole cost matrix have at least one zero element. If not then make the row and /or column with zero element except disturbing the **‘L’-shape**.

20	10	0	60	10
15	4	0	10	0
0	1	0	1	0
5	4	0	4	0
1	0	0	0	0

Step: 4 Procedure of making assignments as follows:

(a) (1) Identify rows successively from top to bottom until a row with exactly one zero element is found. Make an assignment to this single zero by making a square (**☐**) around it. Then cross off(×) all other zeros in the columns.

20	10	$\boxed{0}$	60	10
15	4	$\cancel{0}$	10	0
0	1	$\cancel{0}$	1	0
5	4	$\cancel{0}$	4	0
1	0	$\cancel{0}$	0	0

(2) Identify columns successively from left to right hand with exactly one zero element that has not been assigned. Make assignment to this single zero by making a square (**☐**) around it and hen cross off (×) all other zero element in the corresponding rows.

(b) 1) If a row and/or column has two or more unmarked zeros and one can’t be chosen by inspection then choose zero element arbitrary for assignment.

20	10	$\boxed{0}$	60	10
15	4	$\cancel{0}$	10	$\boxed{0}$
$\boxed{0}$	1	$\cancel{0}$	1	$\cancel{0}$
5	4	$\cancel{0}$	4	$\cancel{0}$
1	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

2) Repeat steps (a) and (b) until one of the following optimality arise.

Step: 5 **<u>Test of Optimality:</u>**

(a) If all the zero elements in the cost matrix are either marked with square (**☐**)
Or are crossed off (×) and there is exactly one assignment in each row and columns then it is an optimal solution. The total cost associated with this solution is obtained by adding the original cost elements in the occupied cells.

20	10	$\boxed{0}$	60	10
15	4	$\cancel{0}$	10	$\boxed{0}$
$\boxed{0}$	1	$\cancel{0}$	1	$\cancel{0}$
5	4	$\cancel{0}$	4	$\cancel{0}$
1	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

Here the 4th row/4th column both doesn’t have the exactly one assignment.

Step: 6 **<u>Revise the optimality cost matrix</u>**

Make set of horizontal and vertical lines to cover all the zeros in the revised cost matrix obtained from step-4, by using the following procedure.

1) Find each row in which the other than maximum zero elements the assignment has made, make a tick (√).

20	10	$\boxed{0}$	60	10	
15	4	$\cancel{0}$	10	$\boxed{0}$	
$\boxed{0}$	1	$\cancel{0}$	1	$\cancel{0}$	✓
5	4	$\cancel{0}$	4	$\cancel{0}$	
1	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	✓

Find each column in which the maximum zero elements other than assignment has made in that column, mark a tick (√)

20	10	0	60	10
15	4	0	10	0
0	1	0	1	0
5	4	0	4	0
1	0	0	0	0

2) Draw a straight lines through each marked column and each marked row.

20	10	0	60	10
15	4	0	10	0
0	1	0	1	0
5	4	0	4	0
1	0	0	0	0

Step: 7develop the new revised opportunity cost matrix.

1) Among the elements in the matrix not covered by any lines, choose the minimum element, define this value as't'.

20	10	0	60	10
15	4	0	10	0
0	1	0	1	0
5	4	0	4	0
1	0	0	0	0

2) Subtract t=4 from every element in the matrix that is not covered by a line.
3) Add t=4 to every element in the matrix covered by the two lines i.e. intersection of two lines.
4) Elements in the matrix covered by one line and '0' are remain unchanged.

16	6	0	56	10
11	0	0	6	0
0	1	4	1	4
1	0	0	0	0
1	0	4	0	4

Step: 8 Repeat steps 4 to 7 until an optimal solution is obtained.

15	6	0	55	9
11	1	1	6	0
0	1	4	0	3
1	1	1	0	0
0	0	4	0	3

The optimal solution is given by:
A→R = 65, B→T = 78, C→ P = 75, D→S = 120, E→ Q = 64
The Total cost = 402

2) **(Maximization problem)** A marketing manager has 5 salesmen and 5 districts. Considering the capabilities of the salesmen and nature of districts, the marketing manager estimates of sales per month (in hundreds of rupees) for each district would be as follows:

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment of salesmen to districts that would result in the maximum sales.

Solution: After applying the new method of an assignment problem we obtained the following optimal solution.

	A	B	C	D	E
1	0	0	11	0	8
2	14	3	1	0	11

3	15	~~0~~	$\boxed{0}$	3	6
4	~~0~~	6	3	4	$\boxed{0}$
5	2	$\boxed{0}$	1	2	2

The optimal solution is given by:
1→A = 32, 2→D = 21, 3→ $C = 33$, 4→E = 36, 5→ B = 33
The Total cost = 155

CONCLUSION:

Here, the technique gives the viable outcome in regards to optimality of a Balanced Assignment issue. Likewise, the aggregate time taken by this strategy for finding optimal solution is less. From this technique we can discover optimal solution or near to the optimal solution of an assignment problem.

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