HIGHER ORDER (b,F,α,β,ρ,d)–CONVEXITY FOR MULTIOBJECTIVE PROGRAMMING PROBLEM WITH SQUARE ROOT TERM

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ABSTRCT

Higher order (b, F, α, β, ρ, d)-convexity is considered. A multiobjective programming problem (MP) in which the numerator and denominator of objective function contain square root of positive semidefinite quadratic form. Mond-Weir and Wolfe type duals are considered for multiobjective programming problem. Duality results are established for multiobjective programming problem under higher order (b, F, α, β, ρ, d)-convexity assumptions. The results are also applied for multiobjective fractional programming problem.

Key Words: Higher order (b, F, α, β, ρ, d)-convexity; Sufficiency; Optimality conditions; Multiobjective Programming; Duality, Multiobjective fractional programming.

1. INTRODUCTION:

Convexity plays an important role in the optimization theory. In inequality constrained optimization the Kuhn-Tucker conditions are sufficient for optimality if the functions are convex. However, the application of the Kuhn-Tucker conditions as sufficient conditions for optimality is not restricted to convex problems as many mathematical models used in decision sciences, economics, management sciences, stochastics, applied mathematics and engineering involve non convex functions.

B-vex functions were introduced by Bector and Singh [1]. Pseudo b-vex and quasi b-vex functions were introduced by Bector et al. [2] and sufficient optimality conditions and duality results for a nonlinear programming problem were obtained under b-vexity conditions on the functions involved. Patel and Naik [3] introduced the concept of b-V-type I functions and their generalizations. Sufficient optimality conditions and duality results were established for multiobjective programming problem for above defined classes of functions.

The concept of (F,ρ)-convexity was considered by Preda [4] as an extension of F-convexity defined by Hanson and Mond [5] and ρ-convexity defined by Vial [6]. Liang et. al. [7] considered a unified formulation of generalized convexity called (F,α,ρ,d)-convexity and obtained some optimality conditions and duality results for nonlinear fractional programming problems. A number of sufficiency theorems for efficient and properly efficient solutions under various generalized convexity assumptions for multiobjective programming problems were obtained by Ahmad [9]. Yuan et. al. [10] introduced the concept of (C,α,ρ,d)-convexity which is the generalization of (F,α,ρ,d)-convexity, and proved optimality conditions and duality theorems for non-differentiable minimax fractional programming problems.

A second order parametric dual for a nondifferentiable minimax fractional programming problem involving square root terms of positive semidefinite quadratic forms was formulated by Ahmad [13] and proved duality results using the concept of second order generalized (F,α,ρ,d)-convexity. Tripathy and Devi [14] introduced a second order multiobjective mixed symmetric duality containing square root term with generalized invex function and established weak duality, strong duality and converse duality theorems under second order (F,ρ)-invexity and (F,ρ)-pseudo invexity assumptions. Tripathy [15] considered a second order duality in multiobjective fractional programming with square root term under generalized univex function and a parameterization technique is used to establish duality results under generalized second order ρ-univexity assumption. Sonali et. al. [16] considered second order duality for minimax fractional programming with square root term involving generalized b-(p, r)-invex functions.

In this chapter, we have considered higher order (b,F,α,β,ρ,d)-convex functions. Under the generalized convexity assumptions, we obtain sufficient optimality conditions for multiobjective programming problem with square root term. Mond-Weir and Wolfe type duals are considered for multiobjective programming problem with square root term. Duality results are established for multiobjective programming problem with square root term under higher order (b,F,α,β,ρ,d)-convexity assumptions. The results are also applied for multiobjective fractional programming problem with square root term.

2. NOTATIONS AND PRELIMINARIES:

We introduce the class of higher order (b,F,α,β,ρ,d)-convex functions as follows:

Let $X \subseteq R^n$ be an open set. Let $f_i : X \to R$, $K: X\times R^n \to R$ be differentiable functions, $F: X\times X\times R^n \to R$ be a sublinear functional in the third variable and $d: X\times X \to R$. Further, let ρ be a real number, $\rho = (\rho^1,\rho^2)$, $\rho^1 = (\rho^1_1,\rho^1_2,\ldots,\rho^1_k) \in R^k$, $\rho^2 = (\rho^2_1,\rho^2_2,\ldots,\rho^2_m) \in R^m$, and $\alpha, \beta: X\times X \to R_+\backslash\{0\}$.

Definition 2.1: The function f_i is said to be higher order $(b,F,\alpha,\beta,\rho^1,d)$-convex at x^0, if for all $x \in X$ and $p \in R^n$,

$$b(x,x^0)[f_i(x) - f_i(x^0)] \geqq F(x,x^0 : \alpha(x,x^0)\{\nabla f_i(x^0)+\nabla_p K(x^0,p)\})$$
$$+ \beta(x,x^0)\{K(x^0,p) - p^T \nabla_p K(x^0,p)\}+ \rho^1 d^2(x,x^0).$$

Remark 2.1: Let $K(x^0, p) = 0$, $b(x,x^0)=1$.

(i) Then the above definition becomes that of (F,α,ρ,d)- convex function introduced by Liang et. al.[7].

(ii) If $\alpha(x,x^0) = 1$, we obtain the definition of (F,ρ)-convex function given by Preda [4].

(iii) If $\alpha(x,x^0) = 1$, $\rho = 0$ and $F(x,x^0; \nabla\phi(x^0)) = \eta^T(x,x^0)\nabla \phi(x^0)$ for a certain map $\eta: X\times X \to R^n$, then (F,α,β,ρ,d)-convexity reduces to the invexity in Hanson [12].

(iv) If F is convex with respect to the third argument, then we obtain the definition of (F,α,ρ,d)-convex function introduced by Yuan et. al. [10].

Remark 2.2: Let $\beta(x,x^0) = 1$, $b(x,x^0)=1$.

(i) If $K(x^0,p)=\frac{1}{2}p^T\nabla^2\phi\left(x^0\right)p$, then the above inequality reduces to the definition of second order (F,α,ρ,d)-convex function given by Ahmad and Husain [11].

(ii) If $\alpha(x, x^0)$ =1, ρ = 0, $K(x^0,p) = \frac{1}{2}p^T\nabla^2\phi(x^0)p$, and $F(x, x^0; a) = \eta^T(x, x^0)a$, where $\eta: X\times X \to R^n$, the above definition becomes that of η-bonvexity introduced by Pandey [18].

We consider the following multiobjective programming problem with square root term:

(MPS) Minimize $f_i(x) = \left[f_1(x)+ (x^TB_1x)^{1/2},\ldots,f_k(x)+(x^TB_kx)^{1/2}\right]$

subject to $h_j(x) \le 0$; $x \in X$,

where X is an open subset of R^n and the functions $f_i:\{f_1,f_2,\ldots,f_k\}:X \to R^k$ and $h_j:\{h_1,h_2,\ldots,h_m\}: X \to R^m$ are differentiable on X. B_i, i=1,2,…,k are positive semi definite matrices of order n. Let U = {$x \in X$: $h_j(x) \leqq 0$} denote the set of all feasible solutions for (MPS).

Proposition 2.1: (Kuhn-Tucker Necessary Optimality Conditions (Mangasarian [17]):

Let $x^0 \in U$ be an optimal solution of (MPS) and let h_j satisfy a constraint qualification. Then there exists a $\mu^0 \in R^k, \lambda^0 \in R^m$ and $w \in R^n$ such that

$$\sum_{i=1}^{k}\mu_i^0[\nabla f_i(x^0) + B_iw] + \sum_{j=1}^{m}\nabla h_j(x^0)\lambda_j^0 = 0, \quad (2.1)$$

$$f_i(x^0) + (x^{0T} B_ix^0)^{1/2} = 0,\ i = 1,2,\ldots k. \quad (2.2)$$

$$\lambda_j^{0T} h_j(x^0) = 0, \quad (2.3)$$

$$(x^{0T}B_ix^0) \le 1,\ i = 1,2,\ldots k. \quad (2.4)$$

$$(x^{0T}B_ix^0)^{1/2} = (x^{0T}B_iw),\ i = 1,2,\ldots k. \quad (2.5)$$

$$\mu_i^0 \ge 0,\ \lambda_j^0 \ge 0,\ h_j(x^0) \le 0, \quad (2.6)$$

Where $\nabla h_j(x^0)$ denotes the $k\times m$ matrix $\left[\nabla h_1(x^0),\nabla h_2(x^0),\ldots,\nabla h_m(x^0)\right]$.

3. SUFFICIENT OPTIMALITY CONDITIONS:

We have established Kuhn-Tucker sufficient optimality conditions for (MPS) under (b,F,α,β,ρ,d)-convexity assumptions.

Theorem 3.1: Let $x^0 \in U, \mu^0 \in R^k$ and $\lambda^0 \in R^m$ satisfy (2.1)-(2.3). If

(i) f_i is higher order (b,F,α,β,ρ^1,d)-convex at x^0,

(ii) $\lambda^{0T}h_j$ is higher order (b,F,α,β,ρ^2,d)-convex at x^0 and

(iii) $\rho^1 + \rho^2 \geqq 0$,

then x^0 is an optimal solution of the problem (MPS).

Proof: Let $x^0 \in U$. Since f_i is higher order (b,F,α,β,ρ^1,d)-convex at x^0, for all $x \in U$, we have

$$b(x, x^0)[[f_i(x)+(x^TB_ix)^{1/2}] - [f_i(x^0)+(x^{0T}B_ix^0)^{1/2}]]$$

$$\geqq F(x, x^0;\alpha(x, x^0)\{\mu_i^0[\nabla f_i(x^0)+B_iw] +\nabla_p K(x^0,p)\})$$

$$+\beta(x,x^0)[K(x^0,p)-p^T\nabla_p K(x^0,p)]+\rho^1 d^2(x,x^0). \quad (3.1)$$

Using (2.1), we get

$$b(x,x^0)[[f_i(x)+(x^TB_ix)^{1/2}] - [f_i(x^0)+(x^{0T}B_ix^0)^{1/2}]]$$

$$\geqq F(x,x^0;\alpha(x,x^0)\{-\nabla h_j(x^0)\lambda_j^0+\nabla_p K(x^0,p)\})$$

$$+\beta(x,x^0)\{K(x^0,p)-p^T\nabla_p K(x^0,p)\}+\rho^1 d^2(x,x^0). \quad (3.2)$$

Also, $\lambda^{0T}h_j$ is higher order (b,F,α,β,ρ^2,d)-convex at x^0. Therefore

$$b(x,x^0)\left[\lambda^{0T}h_j(x) - \lambda^{0T}h_j(x^0)\right]\geqq F(x,x^0;\alpha(x,x^0)[\nabla\lambda^{0T}h_j(x^0)$$

$$-\nabla_p K(x^0,p)])-\beta(x,x^0)\{K(x^0,p)-p^T\nabla_p K(x^0,p)\}+\rho^2 d^2(x,x^0). \quad (3.3)$$

Since $\lambda^{0T}h_j(x^0)=0,\ \lambda^0\geq 0$ and $h_j(x)\leqq 0$, we get

$$0 \geqq F(x,x^0;\alpha(x,x^0)[\nabla\lambda^{0T}h_j(x^0)-\nabla_p K(x^0,p)])$$

$$-\beta(x,x^0)\{K(x^0,p)-p^T\nabla_p K(x^0,p)\}+\rho^2 d^2(x,x^0). \quad (3.4)$$

Adding the inequalities (3.2) and (3.4), we obtain

$$b(x,x^0)\left[[f_i(x)+(x^TB_ix)^{1/2}] - [f_i(x^0)+(x^{0T}B_ix^0)^{1/2}]\right]\geqq(\rho^1+\rho^2)d^2(x,x^0),$$

which by hypothesis (iii) implies,

$$[f_i(x)+(x^TB_ix)^{1/2}] \geqq [f_i(x^0)+(x^{0T}B_ix^0)^{1/2}].$$

Hence x^0 is an optimal solution of the problem (MPS).

4. MOND-WEIR DUALITY:

We have established weak and strong duality theorems for the following Mond-Weir dual (MWMDS) for (MPS):

(MWMDS) Maximize $f_i(u)+(u^TB_iu)^{1/2}$,

$$\text{subject to } \sum_{i=1}^{k}\mu_i[\nabla f_i(u)+B_iw]+\sum_{j=1}^{m}\nabla h_j(u)\lambda_j = 0, \quad (4.1)$$

$$\lambda_j^T h_j(u)\geqq 0,\ j=1,2,\ldots,m \quad (4.2)$$

$$u^TB_iu\leq 1,\ i=1,2,\ldots k.$$

$$(u^TB_iu)^{1/2} = (u^TB_iw),$$

$$u\in X,\ \mu_i\geqq 0,\lambda_j\geqq 0,\ \mu_i\in R^k,\ \lambda_j\in R^m,\ w\in R^n. \quad (4.3)$$

Theorem 8.4.1: (Weak Duality): Let x and (u,μ,λ) be feasible solutions of (MPS) and (MWMDS) respectively. Let

(i) f_i be higher order (b,F,α,β,ρ^1,d)-convex at u,

(ii) λ^Th_j be higher order (b,F,α,β,ρ^2,d)-convex at u, and

(iii) $\rho^1+\rho^2\geqq 0$.

Then

$$[f_i(x)+(x^TB_ix)^{1/2}] \geqq [f_i(u)+(u^TB_iu)^{1/2}].$$

Proof: By hypothesis (i), we have

$$b(x,u)\left[[f_i(x)+(x^TB_ix)^{1/2}] - [f_i(u)+(u^TB_iu)^{1/2}]\right]$$

$$\geqq F(x,u;\alpha(x,u)\{\mu_i[\nabla f_i(u)+B_iw]+\nabla_p K(u,p)\})$$

$$+\beta(x,u)[K(u,p)-p^T\nabla_p K(u,p)]+\rho^1 d^2(x,u). \qquad (4.4)$$

Also hypothesis (ii) yields

$$b(x,u)\left[\lambda^T h_j(x) - \lambda^T h_j(u)\right] \geqq F(x,u;\alpha(x,u)[\nabla\lambda^T h_j(u)$$

$$-\nabla_p K(u,p)]) -\beta(x,u)[K(u,p) - p^T\nabla_p K(u,p)]+ \rho^2 d^2(x,u).$$

By (4.2), (4.3) and $h_j(x) \leqq 0$, it follows that

$$0 \geqq F(x,u;\alpha(x,u)[\nabla\lambda^T h_j(u) - \nabla_p K(u,p)])$$

$$-\beta(x,u)[K(u,p)-p^T\nabla_p K(u,p)]+\rho^2 d^2(x,u). \qquad (4.5)$$

Adding the inequalities (4.4), (4.5) and applying the properties of sublinear functional, we obtain

$$b(x,u)\left[[f_i(x)+(x^TB_ix)^{1/2}] - [f_i(u)+(u^TB_iu)^{1/2}]\right]$$

$$\geqq F(x,u;\alpha(x,u)[\mu_i[\nabla f_i(u)+B_iw]+\nabla\lambda^T h_j(u)])$$

$$+ [\rho^1 d^2(x,u) + \rho^2 d^2(x,u)].$$

which in view of (4.1) implies

$$b(x,u)\left[[f_i(x)+(x^TB_ix)^{1/2}] - [f_i(u)+(u^TB_iu)^{1/2}]\right] \geqq (\rho^1+\rho^2)d^2(x,u).$$

Using hypothesis (iii) in the above inequality, we get

$$[f_i(x)+(x^TB_ix)^{1/2}] \geqq [f_i(u)+(u^TB_iu)^{1/2}].$$

Theorem 4.2: (Strong Duality): Let x^0 be an optimal solution of the problem (MPS) and let h_j satisfy a constraint qualification. Further, let Theorem 4.1 hold for the feasible solution x^0 of (MPS) and all feasible solutions (u,μ,λ) of (MWMDS). Then there exists a $\mu^0 \in R^k, \lambda^0 \in R^m_+$ such that (x^0, μ^0, λ^0) is an optimal solution of (MWMDS).

Proof: Since x^0 is an optimal solution for the problem (MPS) and h_j satisfies a constraint qualification, by Proposition 2.1 there exists a $\mu^0 \in R^k, \lambda^0 \in R^m_+$ such that the Kuhn-Tucker conditions, (2.1)-(2.3) hold. Hence (x^0, μ^0, λ^0) is feasible for (MWMDS).

Now, let (u,μ,λ) be any feasible solution of (MWMDS). Then by weak duality (Theorem 2.1), we have

$$[f_i(x^0)+(x^{0T}B_ix^0)^{1/2}] \geqq [f_i(u)+(u^TB_iu)^{1/2}].$$

Therefore (x^0, μ^0, λ^0) is an optimal solution of (MWMDS).

5. APPLICATION IN MULTIOBJECTIVE FRACTIONAL PROGRAMMING:

The multiobjective programming problem (MPS) becomes the following multiobjective fractional programming problem (MFPS) with square root term:

(MFPS) Minimize $\frac{f_i(x)+(x^TB_ix)^{1/2}}{g_i(x)-(x^TC_ix)^{1/2}}$, i=1,2,…,k.

subject to $\lambda_j^T h_j(x) \leqq 0$, j=1,2,…,m. $x \in X$.

$x^TB_ix \le 1$, i=1,2,…,k.

$(x^TB_ix)^{1/2} = (x^TB_iw)$,

$\lambda_j \geqq 0, \lambda_j \in R^m$.

where f_i, g_i: X→R, $f_i(x) \geqq 0$ and $g_i(x) > 0$ on X and $h_j(x)$, j=1,2,…,m are differentiable functions, B_i and C_i, i =1,2,…,k are positive semi-definite matrices of order n.

We now prove the following result, which gives higher order $(b,F,\alpha^0,\beta^0,\rho,d^0)$-convexity of the ratio function $\frac{f_i(x)+(x^TB_ix)^{1/2}}{g_i(x)-(x^TC_ix)^{1/2}}$.

Theorem 5.1: Let $f_i(x)$ and $-g_i(x)$ be higher order (b,F,α,β,ρ,d)-convex at x^0. Then the multiobjective fractional function $\frac{f_i(x)+(x^TB_ix)^{1/2}}{g_i(x)-(x^TC_ix)^{1/2}}$ is higher-order $(b,F,\alpha^0,\beta^0,\rho,d^0)$-convex at x^0, where

$$\alpha^0(x,x^0) = \alpha(x,x^0)\frac{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}{g_i(x)-(x^TC_ix)^{1/2}},$$

$$\beta^0(x,x^0) = \beta(x,x^0)\frac{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}{g_i(x)-(x^TC_ix)^{1/2}},$$

$$K^0(x^0,p)=\left[\frac{1}{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}+\frac{f_i(x^0)+(x^{0T}B_ix^0)^{1/2}}{[g_i(x^0)-(x^{0T}C_ix^0)^{1/2}]^2}\right]K(x^0,p),$$

$$d^0(x,x^0)=\left[\frac{1}{[g_i(x)-(x^TC_ix)^{1/2}]}+\frac{f_i(x^0)-(x^{0T}B_ix^0)^{1/2}}{[g_i(x)-(x^TC_ix)^{1/2}].[g_i(x^0)-(x^{0T}C_ix^0)^{1/2}]}\right]^{\frac{1}{2}} d(x,x^0).$$

Proof: Since $f_i(x)$ and $-g_i(x)$ are higher order (b,F,α,β,ρ,d)-convex at x^0, we have

$$b(x,x^0)\left[[f_i(x)+(x^TB_ix)^{1/2}] - [f_i(x^0)+(x^{0T}B_ix^0)^{1/2}]\right]$$

$$\geqq F(x,x^0,\alpha(x,x^0)\{[\nabla f_i(x^0)+B_iw]+\nabla_p K(x^0,p)\})$$

$$+\beta(x,x^0)\{K(x^0,p)-p^T\nabla_p K(x^0,p)\}+\rho d^2(x,x^0)$$

and

$$b(x,x^0)\left[-[g_j(x)+(x^TC_jx)^{1/2}]+[g_j(x^0)+(x^{0T}C_jx^0)^{1/2}]\right]$$

$$\geqq F(x,x^0,\alpha(x,x^0)\{-[\nabla g_i(x^0)-C_iw]+\nabla_p K(x^0,p)\})$$

$$+\beta(x,x^0)\{K(x^0,p)-p^T\nabla_p K(x^0,p)\}+\rho d^2(x,x^0).$$

Also

$$b(x,x^0)\left[\frac{f_i(x)+(x^TB_ix)^{1/2}}{g_i(x)-(x^TC_ix)^{1/2}}-\frac{f_i(x^0)+(x^{0T}B_ix^0)^{1/2}}{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}\right]=$$

$$b(x,x^0)\left[\frac{\left[[f_i(x)+(x^TB_ix)^{1/2}]-[f_i(x^0)+(x^{0T}B_ix^0)^{1/2}]\right]}{g_i(x)-(x^TC_ix)^{1/2}}\right.$$

$$\left.+\frac{[f_i(x^0)+(x^{0T}B_ix^0)^{1/2}]\left[-[g_i(x)-(x^TC_ix)^{1/2}]+[g_i(x^0)-(x^{0T}C_ix^0)^{1/2}]\right]}{[g_i(x)-(x^TC_ix)^{1/2}][g_i(x^0)-(x^{0T}C_ix^0)^{1/2}]}\right].$$

Using the above inequalities and sublinearity of F, we get

$$b(x,x^0)\left[\frac{f_i(x)+(x^TB_ix)^{1/2}}{g_i(x)-(x^TC_ix)^{1/2}}-\frac{f_i(x^0)+(x^{0T}B_ix^0)^{1/2}}{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}\right]$$

$$\geqq\frac{1}{g_i(x)-(x^TC_ix)^{1/2}}F(x,x^0;\alpha(x,x^0)\{[\nabla f_i(x^0)+B_iw]+\nabla_pK(x^0,p)\})$$

$$+\frac{1}{g_i(x)-(x^TC_ix)^{1/2}}(\beta(x,x^0)\{K(x^0,p)-p^T\nabla_pK(x^0,p)\}+\rho d^2(x,x^0))$$

$$+\frac{f_i(x^0)+(x^{0T}B_ix^0)^{1/2}}{[g_i(x)-(x^TC_ix)^{1/2}][g_i(x^0)-(x^{0T}C_ix^0)^{1/2}]}F\Big(x,x^0;\alpha(x,x^0)$$

$$\{-[\nabla g_i(x^0)-C_iw]+\nabla_pK(x^0,p)\}\Big)$$

$$+\frac{f_i(x^0)+(x^{0T}B_ix^0)^{1/2}}{[g_i(x)-(x^TC_ix)^{1/2}][g_i(x^0)-(x^{0T}C_ix^0)^{1/2}]}(\beta(x,x^0)\{K(x^0,p)$$

$$-p^T\nabla_pK(x^0,p)\}+\rho d^2(x,x^0)).$$

$$=F(x,x^0;\frac{\alpha(x,x^0)}{g_i(x)-(x^TC_ix)^{1/2}}\{[\nabla f_i(x^0)+B_iw]+\nabla_p K(x^0,p)\})$$

$$+F(x,x^0;\alpha(x,x^0)\frac{f_i(x^0)+(x^{0T}B_ix^0)^{1/2}}{[g_i(x)-(x^TC_ix)^{1/2}][g_i(x^0)-(x^{0T}C_ix^0)^{1/2}]}$$

$$\{-[\nabla g_i(x^0)-C_iw]+\nabla_pK(x^0,p)\}$$

$$+\beta(x,x^0)\left[\frac{1}{[g_i(x)-(x^TC_ix)^{1/2}]}\right.$$

$$+\frac{f_i(x^0)-(x^{0T}B_ix^0)^{1/2}}{[g_i(x)-(x^TC_ix)^{1/2}].[g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}\Bigg]\{K(x^0,p)-p^T\nabla_pK(x^0,p)\}$$

$$+\rho\left[\frac{1}{[g_i(x)-(x^TC_ix)^{1/2}]}+\frac{f_i(x^0)-(x^{0T}B_ix^0)^{1/2}}{[g_i(x)-(x^TC_ix)^{1/2}].[g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}\right]d^2(x,\ x^0).$$

$$=F(x,x^0;\alpha(x,x^0)\frac{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}{g_i(x)-(x^TC_ix)^{1/2}}\Bigg\{\nabla\frac{f_i(x^0)+(x^{0T}B_ix^0)^{1/2}}{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}$$

$$+\left[\frac{1}{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}+\frac{f_i(x^0)+(x^{0T}B_ix^0)^{1/2}}{[g_i(x^0)-(x^{0T}C_ix^0)^{1/2}]^2}\right]\nabla_pK(x^0,p)\Bigg\})$$

$$+\beta(x,\ x^0)\frac{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}{g_i(x)-(x^TC_ix)^{1/2}}\Bigg[\frac{1}{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}$$

$$+\frac{f_i(x^0)+(x^{0T}B_ix^0)^{1/2}}{[g_i(x^0)-(x^{0T}C_ix^0)^{1/2}]^2}\Bigg]\{K(x^0,p)-p^T\nabla_pK(x^0,p)\}$$

$$+\rho\left[\frac{1}{[g_i(x)-(x^TC_ix)^{1/2}]}+\frac{f_i(x^0)-(x^{0T}B_ix^0)^{1/2}}{[g_i(x)-(x^TC_ix)^{1/2}].[g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}\right]d^2(x,\ x^0).$$

Therefore,

$$\left[\frac{f_i(x)+(x^TB_ix)^{1/2}}{g_i(x)-(x^TC_ix)^{1/2}}-\frac{f_i(x^0)+(x^{0T}B_ix^0)^{1/2}}{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}\right]$$

$$\geq F(x,x^0;\alpha^0(x,x^0)\left[\nabla\frac{f_i(x^0)+(x^{0T}B_ix^0)^{1/2}}{g_i(x^0)-(x^{0T}C_ix^0)^{1/2}}+\nabla_pK(x^0,p)\right])$$

$$+\beta^0(x,\ x^0)\{K^0(x^0,p)-p^T\nabla_pK^0(x^0,p)\}+\rho d^2(x,x^0),$$

i.e., $\frac{f_i(x)+(x^TB_ix)^{1/2}}{g_i(x)-(x^TC_ix)^{1/2}}$ is higher order $(b,F,\alpha^0,\beta^0,\rho,d^0)$-convex at x^0.

In view of Theorem 5.1, the results of Section 4 lead to the following duality relations between (MFPS) and its Mond-Weir dual (MWMFDS).

(MWMFDS) Maximize $\frac{f_i(u)+(u^TB_iu)^{1/2}}{g_i(u)+(u^TC_iu)^{1/2}}$

subject to $\sum_{i=1}^{k}\mu_i[[\nabla f_i(u)+B_iw]-v_i[\nabla g_i(u)+C_iw]]+\sum_{j=1}^{m}\nabla h_j(u)\lambda_j=0,$

$\sum_{j=1}^{m}\lambda_j^T\nabla h_j(u)\geqq 0,$

$$u^TB_iu \leq 1, i=1,2,\ldots,k.$$

$$(u^TB_iu)^{1/2} = (u^TB_iw),$$

$$u \in X, \mu_i \geq 0, v_i \geq 0, \lambda_j \geq 0, \mu_i \in R^k, v_i \in R^n, \lambda_j \in R^m.$$

Similar to the proof of Theorems 4.2 and 4.3, we can establish theorems 5.2 and 5.3. Therefore, we simply state them here.

Theorem 5.2: (Weak Duality): Let x and (u,μ,v,λ) be feasible solutions of (MFPS) and (MWMFDS) respectively. Let

(i) f_i and $-g_i$ be higher order $(b,F,\alpha,\beta,\rho^1,d)$-convex at u,

(ii) $\lambda^T h_j$ be higher order $(b,F,\alpha^0,\beta^0,\rho^2,d^0)$-convex at u, where α^0, β^0, K^0 and d^0 are as given in Theorem 5.1, and

(iii) $\rho^1 + \rho^2 \geqq 0$.

Then

$$\frac{f_i(x)+(x^TB_ix)^{1/2}}{g_i(x)-(x^TC_jx)^{1/2}} \geqq \frac{f_i(u)+(u^TB_iu)^{1/2}}{g_i(u)-(u^TB_iu)^{1/2}}.$$

Theorem 5.3: (Strong Duality): Let x^0 be an optimal solution of the problem (MFPS) and let h satisfy a constraint qualification. Further, let Theorem 5.2 hold for the feasible solution x^0 of (MFPS) and all feasible solutions (u,μ,v,λ) of (MWMFDS). Then there exists a $\mu^0 \in R^k, v^0 \in R^k$, $\lambda^0 \in R^m_+$ such that $(x^0, \mu^0, v^0, \lambda^0)$ is an optimal solution of (MWMFDS).

6. WOLFE DUALITY:

The Wolfe dual of (MPS) and (MFPS) are respectively

(WMDS) Maximize $\sum_{i=1}^{k} \mu_i f_i(u) + \sum_{j=1}^{m} \lambda_j{}^T h_j(u)$

subject to $\sum_{i=1}^{k} \mu_i f_i(u) + \sum_{j=1}^{m} \nabla h_j(u)\lambda_j = 0,$

$$u \in X, \mu_i \geq 0, \lambda_j \geq 0, \mu_i \in R^k, \lambda_j \in R^m,$$

(WMFDS) Maximize $\sum_{i=1}^{k} \mu_i \left[f_i(u)-v_i g_i(u)\right] + \sum_{j=1}^{m} \lambda_j{}^T h_j(u)$

subject to $\sum_{i=1}^{k} \mu_i \left[\nabla f_i(u)-v_i \nabla g_i(u)\right] + \sum_{j=1}^{m} \nabla h_j(u)\lambda_j = 0,$

$$u \in X, \mu_i \geq 0, v_i \geq 0, \lambda_j \geq 0, \mu_i \in R^k, v_i \in R^k, \lambda_j \in R^m.$$

Now, we state duality relations for the primal problems (MPS) and (MFPS) and their Wolfe duals (MWMDS) and (WMFDS) respectively. Their proofs follow as in Section 4.

Theorem 6.1: (Weak Duality): Let x and (u,μ,v,λ) be feasible solutions of (MPS) and (MWMDS) respectively. Let

(i) f_i be higher order $(b,F,\alpha,\beta,\rho^1,d)$-convex at u,

(ii) $\lambda^T h_j$ be higher order $(b,F,\alpha,\beta,\rho^2,d)$-convex at u, and

(iii) $\rho^1 + \rho^2 \geqq 0$.

Then

$$f_i(x) \geq f_i(u) + \lambda^T h_j(u).$$

Theorem 6.2: (Strong Duality): Let x^0 be an optimal solution of the problem (MPS) and let h satisfy a constraint qualification. Further, let Theorem 6.1 hold for the feasible solution x^0 of (MPS) and all feasible solutions (u,μ,v,λ) of (WMDS). Then there exists a $\mu^0 \in R^k, v^0 \in R^k$, $\lambda^0 \in R^m_+$ such that $(x^0, \mu^0, v^0, \lambda^0)$ is an optimal solution of (WMDS) and the optimal objective function values of (MPS) and (WMDS) are equal.

Theorem 6.3: (Weak Duality): Let x and (u,μ,v,λ) be feasible solutions of (MFPS) and (WMFDS) respectively. Let

(i) f_i and $-g_i$ be higher order $(b,F,\alpha,\beta,\rho^1,d)$-convex at u,

(ii) $\lambda^T h_j$ be higher order $(b,F,\alpha^0,\beta^0,\rho^2,d^0)$-convex at u, where α^0, β^0, K^0 and d^0 are as given in Theorem 5.1, and

(iii) $\rho^1 + \rho^2 \geqq 0$.

Then $$\frac{f_i(x)}{g_i(x)} \geqq \frac{f_i(u)}{g_i(u)} + \lambda^T h_j(u).$$

Theorem 6.4: (Strong Duality): Let x^0 be an optimal solution of the problem (MFPS) and let h satisfy a constraint qualification. Further, let Theorem 6.3 hold for the feasible solution x^0 of (MFPS) and all feasible solutions (u,μ,v,λ) of (WMFDS). Then there exists a $\mu^0 \in R^k, v^0 \in R^k$, $\lambda^0 \in R^m_+$ such that $(x^0, \mu^0, v^0, \lambda^0)$ is an optimal solution of (WMFDS) and the optimal objective function values of (MFPS) and (WMFDS) are equal.

7. Conclusion

In this paper a new concept of generalized convexity has been introduced. Under this generalized convexity we have established sufficient optimality conditions and duality results for a multiobjective programming problem. These duality relations lead to duality in multiobjective fractional programming.

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