Consistency of Electromagnetic Wave with General Wave Equation in Space and Time

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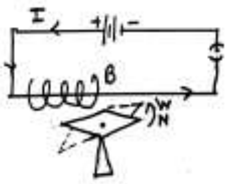
Abstract- Earlier waves were categorised by their dependency on medium and how the medium particles vibrate about their mean position to transfer energy from high to low or zero energy region.
Light had a most common discussion as wave and particle nature. Electromagnetic wave, which does not depends on medium to propagate, can make some effects in antenna or receiver by giving the motion to the charge particles that is electron.
But a wave must have some electric and magnetic effects to start the motion in charge particle. So an electromagnetic wave is created by superimposing the electric and magnetic field vectors.
So in this small paper, Maxwell's equations are discussed, and how electrostatics and magnetism can be put together to make electromagnetism. It is discussed mathematically, how the electric fields and magnetic fields follow the basics of general wave condition in space and time.
Here, the general wave equation is touched with all kinds of wave, no matter about mechanical wave, electromagnetic wave or even light.
Even the starting is done by taking a mechanical wave in string and moved up to light wave, following the consistency of waves with general wave equation in space and time.
So Maxwell put all waves into a single box. And here it is proved mathematically that light is just an electromagnetic wave consisting of electric and magnetic field vectors, mutually perpendicular to each other and perpendicular to the direction of propagation of wave.

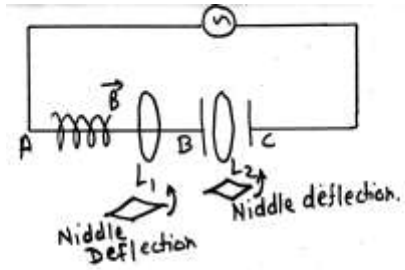
Key words- Maxwell's equations, displacement current, electromagnetic waves, electromagnetism, wave equation in space and time, light, electric and magnetic field vectors.

Introduction-
Maxwell's equations begins with the understanding of the concept that is about Maxwell's displacement current.
So what do one knows and how do one see an electric current?
This is not a kind of visual thing by looking on what one can say this is an electric current.
But one can say about any current carrying wire by observing its effects like just say for example, small child sitting in his home can say about whether the train is coming or going nearby to his home, by just observing an effect, sound, that's it.
So that child must have heard that kind of sound comes by the siren and the engine of train then he learn that these kinds of sound effects are shown by train.
Just like this one got to know some effects shown by a current carrying conductor just like the common example is heating effect due to continuous collision among free electrons and bound electrons in a current carrying conductor.
Similarly the most common effect of an electric current or due to a current carrying conductor is magnetic effect of electric current.



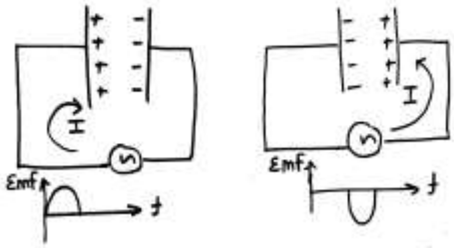
(Figure-1, showing deflection in magnetic compass needle placed near a current carrying conductor)

There must be some change in the position and the direction of magnetic compass needle, as soon as the current starts to flow in wire placed near to a magnetic compass needle. The deflections depends on the magnitude of current every time.
Means again there is some magnetic field which made some effect on magnetic compass needle.
After many experiments, now magnitude and direction of magnetic field can also be shown. So the most visible effect of electric current is magnetic effect due to electric current.
But how the current flows? Due to flow of some charge carrier in conductors say free electrons in metallic conductors?
So accordingly current cannot move in vacuum because of absence of electric charge carrier.



(Figure-2 shows, in section AB current due to flow of charge and hence deflection is observed, which is verified by Amperes Law and some deflection is also found in loop L_2)

In Amperian loop L_1, the deflection in the magnetic compass needle is observed near section AB supporting conventional definition of electric current that is flow of charge carriers (electrons).
But in section BC neither air nor any medium particle is there.
So in case of Amperian loop L_2, there would be no any deflection in the magnetic compass needle as there is no any charge carriers which help to flow of electric current and also by Amperes formula
$\oint \vec{B}.\vec{dl} = \mu\, I_0$, $I_0 = 0$ so B also must be zero.
As in between the plates (across-section BC) no any charge carrier is available to floor no current is moving so line integral of magnetic field must be zero and there would no deflection in the magnetic compass needle.
But it was observed that this time also magnetic needle again showed some deflection,
Means, this time current is due to some other means, accept flow of charge carriers.
These correction was done by Maxwell that not only electrons or charge carrier can create electric current but some other reasons are too.



(Figure-3, shows the generation of time-varying electric field due to applied alternating emf)

Concept of Displacement Current-
In the above figure, the electric field is continuously changing in its magnitude and polarity between the two plates, so this can be the only reason here, for generation of electric current between the plates (in vacuum or free space).
Where the electric field is continuously changing in its magnitude and polarity, just because of alternating emf is provided.

So a varying electric field can also generates an electric current which is known as displacement current, as this displacement current shows all the effects just like conventional conduction current.
During section AB, all the effects are due to conventional conduction current (I_c) that is due to flow of electric charge in material conductor.
But in section BC, a time varying electric field occur in free space between the two plates rather than flow of any particle or charge carriers, called displacement current (I_d).

We know $I_c = \frac{dq}{dt}$, and displacement current is explained by the equation, $I_d = \epsilon_0 \frac{d\Phi e}{dt}$, $\Phi e = \int \vec{E}.\vec{ds}$
Which shows that more the electric field line crossing the plate area, changes, more is the effect shown by electric current, so more will be the magnitude of displacement current.
So everywhere the current I (= $I_c + I_d$) is constant.
During flow of electron, total current was conduction current I_c, as because of absence of time varying electric field, no any displacement current is there during AB, $I_d = 0$, $I = I_c$.
Similarly the current during section BC is because of time varying electric field only, not because of any flow of charged particle, that is only free space is there in between the two plates for Amperian loop L_2.
Maxwell has quite low contribution in making new laws but has a great contribution in corrections and applications of other existing fundamental laws.

Correction done by Maxwell-
The fundamental of electric and magnetic phenomena can be converted into four equations, known as Maxwell's equation.

1. Gauss law in electrostatics-

$$\oiint \vec{E}.\vec{ds} = \frac{\Sigma q}{\epsilon_0}$$

Electric charge is responsible for electric field, no any charge is possible without making electric field or vice-versa and electric field never forms a closed loop due to a single charge.

2. Gauss law in magnetism-

$$\oiint \vec{B}.\vec{ds} = 0$$

Magnetic field always made a closed continuous loop, so whatever lines are entering into a enclosing surface must have to leave the surface or vice versa.

3. Faraday's law of electromagnetic induction-
The emf induce due to change in magnetic flux linked with an area is, $e = -\frac{d\varphi}{Dt}$, here φ is magnetic flux $\vec{B}.\vec{ds}$
But emf induced (e) or potential (V) is related with electric field as, $\int dv = \int \vec{E}.\vec{dl}$
Or, $e = -\frac{d\varphi}{Dt} = \int \vec{E}.\vec{dl} = E\int \vec{dl}$
Or, $$-\frac{d\varphi}{dt} = E\int dl = A\frac{dB}{dt}$$

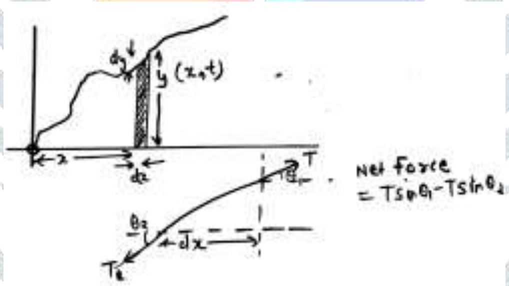
The above equation shows that change in magnetic flux at a location can also create electric field, and this equation joins the characteristics of electrostatics and magnetism from first and second Maxwell equation respectively.

4. Ampere's law-
$\int \vec{B}.\vec{dl} = \mu_0 I = \mu_0 (I_c + I_d) = \mu_0 (I_c + \epsilon_0\frac{d\Phi e}{dt}) = \mu_0 I_c + \mu_0\epsilon_0\frac{d\Phi e}{dt} = \mu_0 I_c + \mu_0\epsilon_0 A\frac{dE}{dt}$
So $$\int \vec{B}.\vec{dl} = \mu_0 I_c + \mu_0\epsilon_0 A\frac{dE}{dt}$$

This equation shows the changing electric field can also generate a magnetic field in space. Anywhere in space and in material that total current I ($=I_c + I_d$) is continuous but displacement current I_d and conventional conduction current I_c individually may not be continuous.

Maxwell's equations in free space-
Now, focusing on free space, where nowhere charge and current is and equation become more symmetric, then all Maxwell's equations can be written as,
(i) $\oiint \vec{E}.\vec{ds} = 0$, (ii) $\oiint \vec{B}.\vec{ds} = 0$, (iii) $-\frac{d\varphi}{Dt} = E\int dl = A\frac{dB}{dt}$ and (iv) $\int \vec{B}.\vec{dl} = \mu_0\epsilon_0 A\frac{dE}{dt}$

For the consistency of electromagnetic wave with general wave equation, all above Maxwell's equations must follow the general wave equation.



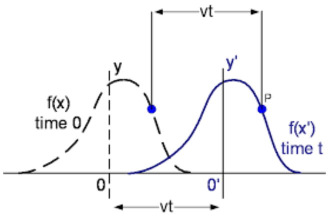
(Figure-4, shows energy moving along x axis by mechanical wave, whose small elemental part dx is picked)

In the diagram, where energy is moving in positive X-direction with the propagation of wave by any material medium. To get the general wave equation, taking a small elemental strip dx from this wave motion,
For small angles, sin Θ ≈ tan Θ,
So net force is T (tan Θ_1 - tan Θ_2)
$T\left(\frac{dy}{dx}\ at\ x + dx - \frac{dy}{dx}\ at\ x\right) = T\frac{\partial^2 y}{\partial x^2}$ dx which is the applied by the left side part of string = $\mu\, dx\, \frac{\partial^2 y}{\partial t^2}$

Here, µ is mass per unit length.

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = 0$$

This above equation shows the origin of wave equation.
If $v = \sqrt{T/\mu}$ that is velocity of wave in string.
So the general wave equation become, $\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$ --------------- (i)
If the wave is propagating in +z-axis with velocity v and displacement in wave at any time is along ± x-axis, then the most general solution of the wave must in the function of (x-vt).
So y can be the solution of the general wave in space and time if y = f(z), where z = x-vt.



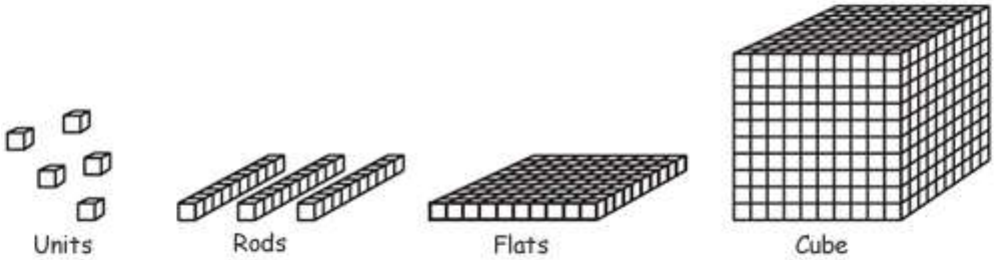
(Figure-5, two bumps (regular and dotted) of waves moving in x direction as the function of x-vt)

Even by this solution, y = f(z), z = x-*v*t, the general wave equation can also be obtain.
Taking partial double derivative of y with respect to x and t. From above equation,
Derivative of y with respect to x,
$\frac{\partial y}{\partial x} = \frac{df}{dz} X \frac{\partial z}{\partial x} = \frac{df}{dz}$, as $\frac{\partial z}{\partial x} = 1$.
By chain rule, taking again derivate,
$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial z^2}$, --------- (ii)
Doing same thing for double derivative of y with respect to t,
$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 f}{\partial z^2} V^2$ or $\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 f}{\partial z^2}$ -------(iii)

From equation (ii) and (iii) we get, $\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$,------(iv) or eq (ii) that is general wave equation in space and time.

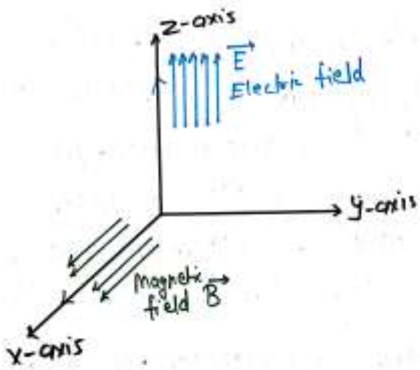
So to satisfy the wave equation, y can be the any function of x±*v*t, for propagation in negative and positive x-axis respectively.

Consistency of electromagnetic wave with general wave equation-
For the consistency of electromagnetic wave with general wave equation, all above Maxwell's equations must follow the general wave equation stated above in eq no. (ii) or (iv), no matter about any loop or enclosed area, of any shape, size or dimension.
If these all four Maxwell's equation works at microscopic level for infinitesimally small loops and elemental areas, they must work for all areas and loops of any shape, size and dimension as small entities combined together to make a big one



(Figure-6, showing any shape is formed by infinitesimally small cubes)

Assuming a wave, consisting of electric and magnetic field vectors, propagating in Y direction, magnetic field vectors and electric field vectors are along X and Z axis respectively.
The displacement of electric and magnetic field vectors in space at any instant of time depends on y and t, as
$\vec{E} = E_z(y,t)\,\hat{k}$, $\vec{B} = B_x(y,t)\,\hat{\imath}$

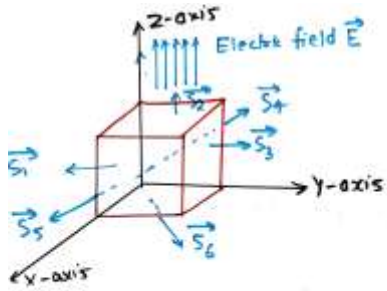


(Figure-7, shows the electric and magnetic field in space)

In most general way, this propagating electromagnetic wave consist of electric and magnetic field vector as the function of space and time, as
$E_z = f(z) = f(y,t)$, $B_x = f(x) = f(y,t)$
And is obtained by getting superimpose two different individual wave fields (say electric and magnetic field vector).

1. Consistency of Maxwell's first and second equation with general wave equation,
Gauss law in electrostatics- $\oiint \vec{E}.\vec{ds} = 0$, Gauss law in magnetism- $\oiint \vec{B}.\vec{ds} = 0$

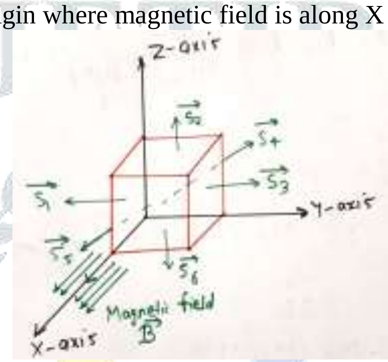
Electric field E and magnetic field B is the function of space and time as E = f(x,y,z,t), B = f′(x,y,z,t), and their double derivative must have to satisfy the general wave equation in space and time.
Assuming an infinitesimal cube placed at the origin where electric field is along Z axis,



(Figure-8, shows electric field crossing few parts of a small cube placed at origin)

Total electric flux passing through this small cube due to electric field acting along Z axis is the algebraic sum of all individual electric flux passing through six faces of the cube,
$\Phi_E = \Phi_{S1} + \Phi_{S2} + \Phi_{S3} + \Phi_{S4} + \Phi_{S5} + \Phi_{S6}$
Electric field is perpendicular to all of the faces except $\vec{S_2}$ And $\vec{S_6}$, so electric no flux is passing through $\vec{S_1}$, $\vec{S_3}, \vec{S_4}$ *and* $\vec{S_5}$ and electric flux passing through surfaces $\vec{S_2}$ And $\vec{S_6}$ are equal and opposite with each other.
No net electric flux is passing through this small cube and Maxwell's first equation is satisfied in the space and time.

Again assuming an infinitesimal cube placed at the origin where magnetic field is along X axis,

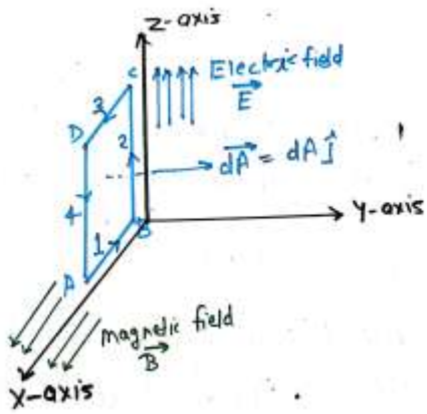


(Figure-9, shows magnetic field crossing few parts of a small cube placed at origin)
Total magnetic flux passing through this small cube due to magnetic field acting X axis is the algebraic sum of all individual magnetic flux passing through six faces of the cube,
$\Phi_B = \Phi_{S1} + \Phi_{S2} + \Phi_{S3} + \Phi_{S4} + \Phi_{S5} + \Phi_{S6}$
Magnetic field is perpendicular to all of the faces except $\vec{S_4}$ And $\vec{S_5}$, so no magnetic flux is passing through $\vec{S_1}$, $\vec{S_2}$, $\vec{S_3}$ And $\vec{S_6}$, magnetic flux passing through surfaces $\vec{S_2}$ and $\vec{S_6}$ are equal and opposite with each other.
No net magnetic flux is passing through this small cube and Maxwell's second equation is also satisfied in the space and time.

But to check the consistency of electromagnetic waves with general wave equation in space and time, the loops also must satisfy these conditions in space and time.

2. Consistency of Maxwell third equation with general wave equation,
   Amperes circuital law, - $\frac{d\varphi}{\mathrm{Dt}}$ = E$\int dl$ or - $\frac{d\varphi}{\mathrm{Dt}}$ = $\int \vec{E}.\vec{dl}$,
   So L.H.S is -$\frac{d\varphi}{\mathrm{Dt}}$ And R.H.S. is $\int \vec{E}.\vec{dl}$

Again assuming an infinitesimal loop in X-Z plain, whose area vector $\vec{dA}$ Is perpendicular to y axis,

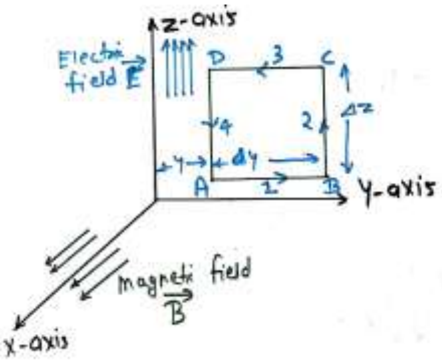


(Figure-10, shows effects of electric and magnetic field crossing an infinitesimal loop in X-Z plain)

Total line integral of electric field through this small loop due to electric field acting along Z axis is the algebraic sum of all individual line integrals through four parts of the loop,

$$\int \vec{E}.\vec{dl} = \int_1 \vec{E}.\vec{dl} + \int_2 \vec{E}.\vec{dl} + \int_3 \vec{E}.\vec{dl} + \int_4 \vec{E}.\vec{dl}$$

Electric field is perpendicular to the 1 ($\overrightarrow{AB}$) and 3 ($\overrightarrow{CD}$), along 2 ($\overrightarrow{BC}$) and opposite to 4 ($\overrightarrow{DA}$), so line integrals of electric field through part 1 ($\overrightarrow{AB}$) and 3 ($\overrightarrow{CD}$), is zero and line integral of electric field through part 2 ($\overrightarrow{BC}$) is just equal and opposite to that of part 4 ($\overrightarrow{DA}$).
So line integral of electric field through the small elemental loop ABCD is zero and R.H.S. become zero.
Area vector of this loop ABCD is perpendicular to magnetic field acting along x axis, so no magnetic flux will pass through this small elemental loop and no change in magnetic flux with respect to time, hence L.H.S. also becomes zero.
This condition is always followed whether the loop is taken in Y-Z plain, X-Y plain or any arbitrary plain in space, so there is no restriction about any plain keeping electric field and magnetic field in their original axes but it is not done until it is verified for loop in Y-Z plain.



(Figure-11, shows effects of electric and magnetic field crossing an infinitesimal loop in Y-Z plain)

Taking $\int \vec{E}.\vec{dl} = -\frac{d\varphi}{\mathrm{Dt}} = -\frac{dB\Delta y\Delta z}{\mathrm{Dt}} = -\Delta y\Delta z\,\frac{dB}{\mathrm{Dt}}$
Where magnetic field is coming out from the centre of the centre of this loop as B.
Taking L.H.S. as $\int \vec{E}.\vec{dl}$ = Δz E (y+Δy) - Δz E y, where AB and CD don't contribute in line integral as both are perpendicular to electric field.
And elements BC and DA are very close to each other so Δz E (y+Δy) - Δz E y can be roughly as $\frac{dE}{\mathrm{Dy}}$-ΔyΔz which also proportional to -ΔyΔz $\frac{dB}{\mathrm{Dt}}$.
Comparing, the final equation is $\frac{dE}{\mathrm{Dy}}\,\Delta y\Delta z = -\Delta y\Delta z\,\frac{dB}{\mathrm{Dt}}$.
And finally got the important relation as $\frac{dE}{\mathrm{Dy}} = -\frac{dB}{\mathrm{Dt}}$---(vi)
And if the loop is taken in X-Y plain the L.H.S. and R.H.S are both going to become zero just like loop made in plain X-Z.

3. Consistency of Maxwell fourth equation with general wave equation-
   $\int \vec{B}.\vec{dl} = \mu_0\epsilon_0 \frac{d\varphi}{dt}$, this can also be verified just like above explanation.

If we take the loop in X-Z plane and Y-Z plane, the L.H.S. and R.H.S. gain comes to zero and if the loop is taken in X-Y plane the final important equation will be $\frac{dB}{dy} = -\mu_0\epsilon_0\,\frac{dE}{dt}$ just like above eq(vi)
The important equations are, $\frac{dE}{\mathrm{Dy}} = -\frac{dB}{\mathrm{Dt}}$ ----(vi) and $\frac{dB}{\mathrm{Dy}} = -\mu_0\epsilon_0\,\frac{dE}{\mathrm{Dt}}$ ----(vii).
Taking derivatives on both sides for the two equation with respect to y,

$$\frac{\partial}{\partial y}\left(\frac{\partial E}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial B}{\partial t}\right) \quad \frac{\partial^2 E}{\partial y^2} = -\frac{\partial^2 B}{\partial y\partial t} = -\frac{1}{\partial t}\left(-\mu_0\epsilon_0\,\frac{dE}{\mathrm{Dt}}\right) = \mu_0\epsilon_0\,\frac{d^2E}{\mathrm{Dt}^2}$$

Or, $\frac{\partial^2 E}{\partial y^2} - \mu_0\epsilon_0\,\frac{d^2E}{\mathrm{Dt}^2} = 0$.
The above equation can be considered as wave equation, this equation give the velocity of wave propagating in y direction and consisting of field vectors.
Comparing above equation with general wave equation $\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$,
$V^2 = \frac{1}{\mu_0\epsilon_0}$, by putting the values of μ_0 and ϵ_0 from magnetism and electrostatics respectively, it become c (=3x10^8m/s) that is velocity of light in vacuum.

Conclusion about light by Maxwell's equations- A new conclusion can also be made as light is just an electromagnetic wave consisting of electric and magnetic field vectors, mutually perpendicular to each other and perpendicular to the direction of propagation of wave. Where the electric field and magnetic field keep on generating each other. In case if E changes and become totally vanish this field gets converted into magnetic field and similarly for magnetic field. So these fields are self-sustaining in space and keep propagating in space by oscillating their values.
So here electricity, magnetism and light can be put into a single box.

Till now it is clear that magnetic field and electric field vectors oscillate and change their value from zero to maximum value. So field vectors can be considered as sine function or cosine function of displacement vector along their respective axes.

As, $\vec{E} = \vec{E_0}$ Sin(ky-ωt) $\hat{k}$ And $\vec{B} = \vec{B_0}$ Sin(ky-ωt) $\hat{\imath}$,

Using, $\frac{dE}{Dy} = -\frac{dB}{Dt}$ and $\frac{dB}{Dy} = -\mu_0\epsilon_0\frac{dE}{Dt}$.--- (viii)

Now to obey the wave equation, above field function must obey these equations.

K E_0 cos(ky- ωt) = - B_0 (-ω) cos(ky-ωt) = B_0 ω

Or, $E_0 = \frac{\omega}{K} B_0$ ---------- (ix)

Using ---- (viii), B_0 k cos(ky-ωt) = - E_0 $\mu_0\epsilon_0$ (-ω) cos(ky-ωt),

Or, B_0 k = $\mu_0\epsilon_0$ (ω) $E_0 = \frac{\omega}{C^2}$ ----------------(x)

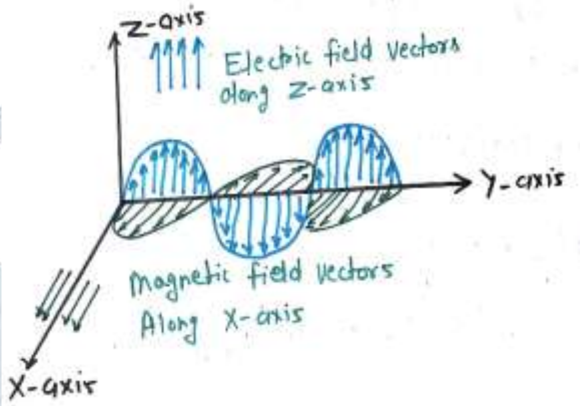
Comparing the above the equations (ix) and (x), ω = kc.

Putting the value of ω into electric or magnetic field vector

Sin(ky- ωt) = Sin(ky- kct) = Sink(y- ct), so these become the function of y-vt or y-ct. Hence the general wave equation is again satisfied.

And now also, $E_0 = cB_0$

Which shows that the maximum value of electric field vector is c(=3x10^{8}m/s) times the maximum value of magnetic field vector.



(Figure-12 shows, electromagnetic wave propagating along positive Y-axis.)

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