A NOTE ON THE TRACE INEQUALITY FOR PRODUCTS OF QUATERNION HERMITIAN MATRIX POWER

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Abstract:

Da-Wei Zhang [3] obtained the inequality $tr(AB)^{2^k} \le trA^{2^k}B^{2^k}$ for Hermitian matrices A and B, where k is natural number. Hence it is proved that these results hold when the power index of the product of quaternion Hermitian Matrices A and B is a non negative even number. In the mean time, it is pointed out that the relation between $tr(AB)^m$ and trA^mB^m is complicated when the power index m is a non negative odd number, therefore the above inequality cannot be generalized to all non negative integers. As an application, we not only improve the results of Xiaojing Yang [J.M.A.A., 250, 372-374] Xinmin Yang [J.M.A.A.,263,2001:327-333] and Fozi M. Dannan [J.Ineq.Pure and Appl.Math.,2(3)(2001), Art.34], but also give the complete resolution for the question of the trace inequality about the powers of quaternion Hermitian and Skew Hermitian matrices that is proposed by Zhengming Jiao.

Key words:

Hermitian matrix, Quaternion Hermitian matrix, Trace, Inequality, Skew Hermitian matrix.

1.Introduction:

Let $H^{n\times n}$ be the set of all $n \times n$ quaternion matrices over the field $\mathbb{H}$. The modulus of all diagonal entries of the quaternion matrix $A = (a_{ij}) \in H^{n\times n}$ are arranged in decreasing order as $|\delta_1(A)| \ge |\delta_2(A)| \ge \cdots \ge |\delta_n(A)|$, i.e, $\delta_1(A), \delta_2(A), \dots, \delta_n(A)$ is an entire arrangement of $a_{11}, a_{22}, \dots, a_{nn}$ all its singular values satisfy $\sigma_1(A) \ge \sigma_2(A) \ge \cdots \ge \sigma_n(A)$. In particular, when the eigenvalues of A are real numbers, let its eigenvalues satisfy $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$; A^H, trA denotes its conjugate transpose matrix and trace respectively. Further, let $H(n), H_0^+(n), H^+(n), HS(n)$ be the subsets of all quaternion Hermitian, quaternion Hermitian semi positive definite, quaternion Hermitian positive definite and quaternion skew Hermitian matrices. Finally, let $A^{\frac{1}{2}}$ represent the quadratic root of $A \in H_0^+(n)$ and $\mathbb{R}, \mathbb{N}$ denote the sets of all real numbers and non negative integers. The complex number $\sqrt{-1} \in C$ satisfies $\left(\sqrt{-1}\right)^2 = -1$

2. Some Lemmas:

Lemma 2.1:

Let $A, B \in H^{n\times n}$, then

$$\sum_{i=1}^{t}|\delta_i((AB)^m)| \le \sum_{i=1}^{t}\lambda_i((A^HABB^H)^m)$$
$$\le \sum_{i=1}^{t}\lambda_i((A^HA)^m(BB^H)^m), 1 \le t \le n, m \in \mathbb{N} \quad \dots (2.1)$$

Proof:

Let $A = A_0 + A_1j + A_2k, B = B_0 + B_1j + B_2k$, $AB = A_0B_0 + A_1B_1j + A_2B_2k$ and $(AB)^m = (A_0B_0)^m + (A_1B_1)^mj + (A_2B_2)^mk$

$|\delta_i((AB)^m)| = |\delta_i(A_0B_0)^m + \delta_i(A_1B_1)^mj + \delta_i(A_2B_2)^mk|$

$\sum_{i=1}^{t}|\delta_i((AB)^m)| = \sum_{i=1}^{t}[|\delta_i(A_0B_0)^m| + |\delta_i(A_1B_1)^mj| + |\delta_i(A_2B_2)^mk|]$

$\le \sum_{i=1}^{t}\lambda_i(A_0^HA_0B_0B_0^H)^m + \sum_{i=1}^{t}\lambda_i(A_1^HA_1B_1B_1^H)^m j + \sum_{i=1}^{t}\lambda_i(A_2^HA_2B_2B_2^H)^m k$

[Since Lemma 2.1]

$\le \sum_{i=1}^{t}\lambda_i(A_0^HA_0)^m(B_0B_0^H)^m + \sum_{i=1}^{t}\lambda_i(A_1^HA_1)^m(B_1B_1^H)^m + \sum_{i=1}^{t}\lambda_i(A_2^HA_2)^m(B_2B_2^H)^m$

[Since Lemma 2.1]

$= \sum_{i=1}^{t}\lambda_i^m(A_0^H)\lambda_i^m(A_0)\lambda_i^m(B_0)\lambda_i^m(B_0^H) + \sum_{i=1}^{t}\lambda_i^m(A_1^H)\lambda_i^m(A_1)\lambda_i^m(B_1)\lambda_i^m(B_1^H)j + \sum_{i=1}^{t}\lambda_i^m(A_2^H)\lambda_i^m(A_2)\lambda_i^m(B_2)\lambda_i^m(B_2^H)k$

$\le \sum_{i=1}^{t}\lambda_i(A_0^{H^m})\lambda_i(A_0^m)\lambda_i(B_0)\lambda_i(B_0^{H^m}) + \sum_{i=1}^{t}\lambda_i(A_1^{H^m})\lambda_i(A_1^m)\lambda_i(B_1)\lambda_i(B_1^{H^m})j + \sum_{i=1}^{t}\lambda_i(A_2^{H^m})\lambda_i(A_2^m)\lambda_i(B_2)\lambda_i(B_2^{H^m})k$

$= \sum_{i=1}^{t}\lambda_i(A_0^{H^m}A_0^m)(B_0B_0^{H^m}) + \sum_{i=1}^{t}\lambda_i(A_1^{H^m}A_1^m)(B_1B_1^{H^m})j + \sum_{i=1}^{t}\lambda_i(A_2^{H^m}A_2^m)(B_2B_2^{H^m})k$

$\le \sum_{i=1}^{t}\lambda_i((A^HABB^H)^m)$ [Since Lemma 2.1]

Therefore, $\sum_{i=1}^{t}|\delta_i((AB)^m)| \le \sum_{i=1}^{t}\lambda_i((A^HABB^H)^m)$ … (1)

$\sum_{i=1}^{t}\lambda_i((A^HABB^H)^m) = \sum_{i=1}^{t}\lambda_i((A_0^HA_0B_0B_0^H + A_1^HA_1B_1B_1^Hj + A_2^HA_2B_2B_2^Hk)^m)$

$= \sum_{i=1}^{t}\lambda_i(A_0^HA_0B_0B_0^H)^m + \sum_{i=1}^{t}\lambda_i(A_1^HA_1B_1B_1^H)^m j + \sum_{i=1}^{t}\lambda_i(A_2^HA_2B_2B_2^H)^m k$

$\le \sum_{i=1}^{t}\lambda_i(A_0^HA_0)^m(B_0B_0^H)^m + \sum_{i=1}^{t}\lambda_i(A_1^HA_1)^m(B_1B_1^H)^m j + \sum_{i=1}^{t}\lambda_i(A_2^HA_2)^m(B_2B_2^H)^m k$

[Since Lemma 2.1]

$= \sum_{i=1}^{t}\lambda_i^m(A_0^H)\lambda_i^m(A_0)\lambda_i^m(B_0)\lambda_i^m(B_0^H) + \sum_{i=1}^{t}\lambda_i^m(A_1^H)\lambda_i^m(A_1)\lambda_i^m(B_1)\lambda_i^m(B_1^H)j + \sum_{i=1}^{t}\lambda_i^m(A_2^H)\lambda_i^m(A_2)\lambda_i^m(B_2)\lambda_i^m(B_2^H)k$

$\le \sum_{i=1}^{t}\lambda_i(A_0^{H^m})\lambda_i(A_0^m)\lambda_i(B_0)\lambda_i(B_0^{H^m}) + \sum_{i=1}^{t}\lambda_i(A_1^{H^m})\lambda_i(A_1^m)\lambda_i(B_1)\lambda_i(B_1^{H^m})j + \sum_{i=1}^{t}\lambda_i(A_2^{H^m})\lambda_i(A_2^m)\lambda_i(B_2)\lambda_i(B_2^{H^m})k$

$\le \sum_{i=1}^{t}\lambda_i((A^HABB^H)^m)$

$\le \sum_{i=1}^{t}\lambda_i((A^HA)^m(BB^H)^m)$

Therefore, $\sum_{i=1}^{t}\lambda_i((A^HABB^H)^m) \le \sum_{i=1}^{t}\lambda_i((A^HA)^m(BB^H)^m)$ … (2)

Equation (1) and (2) gives,

$$\sum_{i=1}^{t}|\delta_i((AB)^m)| \le \sum_{i=1}^{t}\lambda_i((A^HABB^H)^m)$$
$$\le \sum_{i=1}^{t}\lambda_i((A^HA)^m(BB^H)^m)$$

The proof is completed.

Lemma 2.2:

Let $A = (a_{ij}), B = (b_{ij}) \in H(n)HS(n)$, then $tr\ A * B \in \mathbb{R}$.

Proof:

Let $A = A_0 + A_1j + A_2k, B = B_0 + B_1j + B_2k$

when $A, B \in H(n)$. It is known that $tr\ A * B \in \mathbb{R}$.

By the simple fact, $F \in HS(n)$ if and only if $\sqrt{-1}F \in H(n)$, it follows that $\sqrt{-1}(A_0 + A_1j + A_2k), \sqrt{-1}(B_0 + B_1j + B_2k) \in H(n)$ holds when $A, B \in HS(n)$. Thus from the proved result, at this time, it is easy to know,

$$tr\ A * B = tr\left(-\left(\sqrt{-1}(A_0 + A_1j + A_2k)\right) * \left(\sqrt{-1}(B_0 + B_1j + B_2k)\right)\right)$$
$$= -tr\left(\sqrt{-1}A\right) * \left(\sqrt{-1}B\right) \in \mathbb{R}$$

Therefore, $tr\ A * B \in \mathbb{R}$

The proof is completed.

Lemma 2.3:

Let $A, B \in H_0^+(n)$, then

(i) $0 \le trAB \le tr\ A\ trB$ and **(ii)** $0 \le trA^m \le (tr\ A)^m, m \in \mathbb{N}$

Proof:

(i) Let $A, B \in H_0^+(n)$ then we prove $0 \le trAB \le tr\ A\ trB$

Here, $A = A_0 + A_1j + A_2k, B = B_0 + B_1j + B_2k, AB = A_0B_0 + A_1B_1j + A_2B_2k$

$$tr\ AB = tr(A_0B_0 + A_1B_1j + A_2B_2k)$$
$$= tr(A_0B_0) + tr(A_1B_1)j + tr(A_2B_2)k$$
$$\le tr(A_0)tr(B_0) + tr(A_1)tr(B_1)j + tr(A_2)tr(B_2)k$$
$$\le tr\ A\ trB$$

Therefore, $trAB \le tr\ A\ trB$

(ii) We prove $0 \le trA^m \le (tr\ A)^m, m \in \mathbb{N}$

Since, $A = A_0 + A_1j + A_2k, A^m = A_0^m + A_1^mj + A_2^mk$

$$A^m = A_0 \ldots m\ times + A_1j \ldots m\ times + A_2k \ldots m\ times$$
$$trA^m = tr[A_0 \ldots m\ times + A_1j \ldots m\ times + A_2k \ldots m\ times]$$
$$= tr(A_0)^m + tr(A_1)^mj + tr(A_2)^mk$$
$$\le tr\ A_0^m + tr\ A_1^mj + tr\ A_2^mk$$
$$= tr[A_0^m + A_1^mj + A_2^mk]$$
$$\le tr(A_0 + A_1j + A_2k)$$

$\leq (trA)^m$

Therefore, $trA^m \leq (tr\ A)^m$

The proof is completed.

Lemma 2.4:

Let $A, B \in H(n)$, then $tr(AB)^m, trA^mB^m \in \mathbb{R}$ for all $m \in N$.

Proof:

Let $A, B \in H(n)$, here, $A = A_0 + A_1j + A_2k, B = B_0 + B_1j + B_2k$

$(AB)^m = (A_0B_0)^m + (A_1B_1)^mj + (A_2B_2)^mk$

$(AB)^m = A_0B_0 \dots m\ times + A_1B_1j \dots m\ times + A_2B_2k \dots m\ times$

$tr\ (AB)^m = tr(A_0B_0 \dots m\ times + A_1B_1j \dots m\ times + A_2B_2k \dots m\ times)$

$= tr(A_0B_0)^m + tr(A_1B_1)^mj + tr(A_2B_2)^mk$

$\leq trA_0^m trB_0^m + trA_1^m trB_1^m j + trA_2^m trB_2^m k$

$= tr[A_0^mB_0^m + A_1^mB_1^mj + A_2^mB_2^mk]$

$\leq tr(A^mB^m) \in \mathbb{R}$

$\leq tr(A^m)tr(B^m) \in \mathbb{R}$ for all $m \in N$

Therefore, $tr(AB)^m \leq tr(A^m)tr(B^m) \in \mathbb{R}$ for all $m \in N$

Now, $trA^mB^m = tr(A_0B_0 \dots m\ times + A_1B_1j \dots m\ times + A_2B_2k \dots m\ times\ )$

$= tr(A_0^mB_0^m + A_1^mB_1^mj + A_2^mB_2^mk)$

$\leq tr(A_0^mB_0^m) + tr(A_1^mB_1^m)j + tr(A_2^mB_2^m)k$

$\leq tr(A^mB^m)$

$\leq tr(AB)^m$

Therefore, $trA^mB^m \leq tr(AB)^m \in \mathbb{R}$ for all $m \in N$

The proof is completed.

Lemma 2.5:

Let $\mathrm{A} \in \mathrm{H(n)}, \mathrm{B} \in \mathrm{HS(n)}, \mathrm{m} \in \mathbb{N}$, then

$$tr(AB)^m = \left(-\sqrt{-1}\right)^m tr\left(A\left(\sqrt{-1}B\right)\right)^m \qquad \dots (1)$$

$trA^mB^m = \left(-\sqrt{-1}\right)^m trA^m\left(\sqrt{-1}B\right)^m$ and for $m = 2t(t \in N), tr(AB)^m, trA^mB^m$ are all real. Further, when $m = 2t + 1(t \in N), tr(AB)^m, trA^mB^m$ are all zeroes or pure imaginary numbers.

Proof:

Without loss of generality, assume that $m \geq 2$, similarly

$$tr(AB)^m = tr\left(A_0 + A_1j + A_2k\left(-\sqrt{-1}\left(\sqrt{-1}B_0 + B_1j + B_2k\right)\right)\right)^m$$

$$= \left(-\sqrt{-1}\right)^m tr\left(A_0 + A_1j + A_2k\left(\sqrt{-1}B_0 + B_1j + B_2k\right)\right)^m$$

$$= \left(-\sqrt{-1}\right)^m tr\left(A\left(\sqrt{-1}B\right)\right)^m$$

$$trA^mB^m = trA^m tr\left(-\sqrt{-1}\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)\right)^m$$

$$= \left(-\sqrt{-1}\right)^m trA_0^m + A_1^m j + A_2^m k\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)^m$$

So that (1) holds. When $m = 2t(t \in \mathbb{N}), \left(-\sqrt{-1}\right)^m = (-1)^{3t} \in \mathbb{R}$, thus by (1) and Lemma 2.4, one obtains that $tr(AB)^m, trA^mB^m$ are all real.

When $m = 2t + 1(t \in \mathbb{N}), \left(-\sqrt{-1}\right)^m = (-1)^{3t+1}\sqrt{-1} \notin \mathbb{R}$, then we have that $tr(AB)^m$, trA^mB^m are all zeroes or pure imaginary numbers by Lemma 2.5 and Lemma 2.4.

The proof is completed.

Lemma 2.6:

Let $A, \mathrm{B} \in \mathrm{HS(n)}, \mathrm{m} \in \mathrm{N}$, then

$$tr(AB)^m = (-1)^m tr\left(\left(\sqrt{-1}A\right)\left(\sqrt{-1}B\right)\right)^m$$

$$trA^mB^m = (-1)^m tr\left(\sqrt{-1}A\right)^m\left(\sqrt{-1}B\right)^m \in \mathbb{R} \quad \text{.... (1)}$$

Proof:

Let $A, \mathrm{B} \in \mathrm{HS(n)}, \mathrm{m} \in \mathbb{N}$, here, $A = A_0 + A_1 j + A_2 k, B = B_0 + B_1 j + B_2 k,$

$$AB = A_0B_0 + A_1B_1 j + A_2B_2 k$$

$$tr(AB)^m = tr(A_0B_0 + A_1B_1 j + A_2B_2 k)^m$$

$$= tr(A_0^mB_0^m + A_1^mB_1^m j + A_2^mB_2^m k)$$

$$\leq tr(A_0^mB_0^m) + tr(A_1^mB_1^m)j + tr(A_2^mB_2^m)k$$

$$tr(AB)^m \leq tr(A_0^m)tr(B_0^m) + tr(A_1^m)tr(B_1^m)j + tr(A_2^m)tr(B_2^m)k$$

$$\leq trA^m trB^m$$

$$tr(AB)^m = (-1)^m tr\left(\left(\sqrt{-1}A\right)\left(\sqrt{-1}B\right)\right)^m$$

$$= (-1)^m tr\left(\left(\sqrt{-1}A_0 + A_1 j + A_2 k\right)\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)\right)^m$$

$$tr(AB)^m = (-1)^m tr\left(\left(\sqrt{-1}A\right)^m\left(\sqrt{-1}B\right)^m\right) \in \mathbb{R}$$

Now, $trA^mB^m = tr(A_0^mB_0^m + A_1^mB_1^m j + A_2^mB_2^m k)$

$$\leq tr(A_0^mB_0^m) + tr(A_1^mB_1^m)j + tr(A_2^mB_2^m)k$$

$$\leq tr(A_0^m)tr(B_0^m) + tr(A_1^m)tr(B_1^m)j + tr(A_2^m)tr(B_2^m)k$$

$$\leq tr(AB)^m$$

$$trA^mB^m = (-1)^m tr\left(\sqrt{-1}A_0 + A_1 j + A_2 k\right)^m\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)^m \in \mathbb{R}$$

$$= (-1)^m tr\left(\sqrt{-1}A_0^m + A_1^m j + A_2^m k\right)\ \left(\sqrt{-1}B_0^m + B_1^m j + B_2^m k\right) \ \in \mathbb{R}$$

$trA^mB^m = (-1)^m tr\left(\sqrt{-1}A\right)^m\left(\sqrt{-1}B\right)^m \in \mathbb{R}$. We have that $tr(AB)^m, trA^mB^m$ are all zeros or pure imaginary numbers by (1) and Lemma 2.5.

The proof is completed.

3.Main Results:

Theorem 3.1

Let A, B, ∈ $H^{n\times n}$, then

(i) $|tr(AB)^{2m}| \leq tr(A^H ABB^H)^m \leq tr(A^H A)^m (BB^H)^m, m \in \mathbb{N}.$ … (3.1)

(ii) $|tr(AB)^{2m}| \leq tr(A^H ABB^H)^m$

$\leq tr(A^H A)^m (BB^H)^m$

$\leq tr\ ((A^H A)^{\frac{1}{2}})^2 tr\ (A^H A)^{m-1} tr(BB^H)^m$

$\leq (tr(A^H A)^{\frac{1}{2}})^2 (trA^H A)^{m-1} tr(BB^H)^m, m(\geq 1) \in \mathbb{N}$ … (3.2)

Proof:

(i) Take $t = n$ in (2.1), then we have that

$|tr(AB)^{2m}| = |\sum_{i=1}^{n} \delta i((AB)^{2m})|$

$=|\sum_{i=1}^{n} \delta i((A_0B_0 + A_1B_1j + A_2B_2k)^{2m})|$

$\leq \sum|\delta i((A_0B_0 + A_1B_1j + A_2B_2k)^{2m})|$

$\leq \sum_{i=1}^{n} \lambda_i((A_0^H A_0 B_0 B_0^H + A_1^H A_1 B_1 B_1^H j + A_2^H A_2 B_2 B_2^H k)^m)$

$\leq \sum_{i=1}^{n} \lambda_i\ ((A_0^H A_0 B_0 B_0^H)^m + (A_1^H A_1 B_1 B_1^H)^m j + A_2^H A_2 B_2 B_2^H)^m k)$

$\leq \sum_{i=1}^{n} \lambda_i((A^H ABB^H)^m)$

$=\ tr(A^H ABB^H)^m$

$\leq \sum_{i=1}^{n} \lambda_i\ ((A^H A)^m (BB^H)^m) = tr((A^H A)^m (BB^H)^m)$

The proof of (i) is completed.

(ii) giving (3.1)

By $A^H A, BB^H \in H_0^+(n)$ and Lemma (2.3), When $m \geq 1$ one can we get

$tr(A^H A)^m (BB^H)^m \leq tr[(A_0^H A_0 + A_1^H A_1 j + A_0^H A_0 k)(A_0^H A_0 + A_1^H A_1 j + A_2^H A_2 k)^{m-1}]$

$tr(B_0 B_0^H A)(B_1 B_1^H j + B_1 B_1^H k)^m$

$\leq tr(A^H A)(A^H A)^{m-1} tr(BB^H)^m$

$\leq tr((A_0^H A_0 + A_1^H A_1 j + A_2^H A_2 k)^{\frac{1}{2}})^2 tr(A_0^H A_0 + A_1^H A_1 j + A_2^H A_2 k)^{m-1} (trB_0 B_0^H + B_1 B_1^H j$

$+B_2 B_2^H k)^m$

$\leq (tr(A^H A)^{\frac{1}{2}})^2 tr(A^H A)^{m-1} tr(BB^H)^m$

$\leq tr(A_0^H A_0 + A_1^H A_1 j + A_2^H A_2 k)^{\frac{1}{2}}))^2 tr(A_0^H A_0 + A_1^H A_1 j + A_2^H A_2 k)^{m-1}\ (trB_0 B_0^H + B_1 B_1^H j$

$+B_2 B_2^H k)^m$

$\leq (tr(A^H A)^{\frac{1}{2}})^2 tr(A^H A)^{m-1} (trBB^H)^m$

Therefore (3.2) is correct by (3.1).

The proof is completed.

Theorem 3.2

Let $A, B \in H(n)$ then

$tr(AB)^{2m} \leq |tr(AB)^{2m}| \leq tr(A^2B^2)^m \leq trA^{2m}B^{2m}, m \in \mathbb{N}$. …. (3.3)

Proof:

Let $A = A_0 + A_1j + A_2k,\ B = B_0 + B_1j + B_2k$ and $AB = A_0B_0 + A_1B_1j + A_2B_2k$

From Lemma(2.4), it is obtained that $tr(AB)^m$, trA^mB^m are all real, then by

$tr\ (AB)^{2m} \leq |tr(AB)^{2m}|$

$tr\ (AB)^{2m} = tr(A_0B_0 + A_1B_1j + A_2B_2k)^{2m}$

$= |tr\ (A_0B_0 + A_1B_1j + A_2B_2k)^{2m}\ |$

$\leq |tr\ (A_0B_0)^{2m}| + |tr\ (A_1B_1)^{2m}j| + |tr(A_2B_2)^{2m}k\ |$

$\leq |tr(AB)^{2m}|$

Therefore, $tr(AB)^{2m} \leq |tr(AB)^{2m}|$ …. (1)

Applying $A^H = A, B^H = B$ in (3.1), we have

$|tr(AB)^{2m}| \leq tr(A^HABB^H)^m \leq tr(AABB)^m \leq tr(A^2B^2)^m$

Therefore, $|tr(AB)^{2m}| \leq tr(A^2B^2)^m$ …. (2)

Now, $tr(A^2B^2)^m \leq tr(A_0^2B_0^2 + A_1^2B_1^2j + tr(A_2^2B_2^2k)^m$

$\leq tr(A_0^2B_0^2)^m + tr(A_0^2B_0^2)^mj + tr(A_2^2B_2^2)^mk$

$\leq tr((AB)^2)^m$

$\leq tr(A^{2m}B^{2m})$

Therefore, $tr(A^2B^2)^m \leq\ tr(A^{2m}B^{2m})$ … (3)

Combining (1), (2) and (3), we get,

$tr(AB)^{2m} \leq |tr(AB)^{2m}| \leq tr(A^2B^2)^m \leq\ trA^{2m}B^{2m}, m \in \mathbb{N}$

The proof is completed.

Note:

The trace inequality of two powered quaternion hermitian matrices as follows.

$tr(AB)^{2^k} \leq trA^{2^k}B^{2^k}, A, B \in H(n), k \in \mathbb{N}$ can be achieved by (3.3). When $A, B \in H_0^+(n)$, the inequality $tr(AB)^m \leq tr\ A^mB^m, m \in \mathbb{N}$ is obtained by replacing A, B if (3.3) with $A^{\frac{1}{2}}, B^{\frac{1}{2}}$.

The above procedure indicates we can give a simple proof for the inequality

$tr(AB)^m \leq trA^mB^m, m \in \mathbb{N}$.

Theorem:3.3

Let $A, B \in H(n), m \in \mathbb{N}, then$

(i) $tr(AB)^{2m} \leq tr(A^2B^2)^m \leq trA^{2m}B^{2m} \leq (trA^{4m})^{\frac{1}{2}}(trB^{4m})^{\frac{1}{2}}$ … (3.4)

(ii) $tr(AB)^{2m} \leq tr(A^2B^2)^m \leq tr(A^2B^2)^m$ … (3.5)

(iii) $tr(AB)^{2m} \leq tr(A^2B^2)^m \leq trA^{2m}B^{2m}$

$\leq trA^2trA^{2(m-1)}trB^{2m}$

$$\leq (tr(A^2)^{\frac{1}{2}})^2(trA^2)^{m-1}(trB^2)^m, when\ m \geq 1 \quad \ldots (3.6)$$

Proof:

Let $A, B \in H(n), m \in \mathbb{N}$

Here $A = A_0 + A_1j + A_2k,\ B = B_0 + B_1j + B_2k\ \ AB = A_0B_0 + A_1B_1j + A_2B_2k$

(i) $tr(AB)^{2m} = tr(A_0B_0 + A_1B_1j + A_2B_2k)^{2m}$

$= tr\ (A_0B_0)^{2m} + tr(A_1B_1)^{2m}j + tr(A_2B_2)^{2m}k$

$\leq\ tr(A_0^2B_0^2)^m + tr(A_1^2B_1^2)^mj + tr(A_2^2B_2^2)^mk \leq tr(A^2B^2)^m$

Therefore, $tr(AB)^{2m}= tr(A^2B^2)^m$ …(1)

$tr(A^2B^2)^m = tr(A_0^2B_0^2 + A_1^2B_1^2j + A_2^2B_2^2k)^m$

$\leq tr\ (A_0^2B_0^2)^m + tr(A_1^2B_1^2)^mj + tr(A_2^2B_2^2)^mk$

$\leq tr(A_0^{2m}B_0^{2m}) + tr(A_1^{2m}B_1^{2m})j + tr(A_2^{2m}B_2^{2m})k \leq\ tr(A^{2m}B^{2m})$

$\mathrm{tr}(A^2B^2)^m \leq trA^{2m}B^{2m}$ … (2)

$trA^{2m}B^{2m} = trA_0^{2m}B_0^{2m} + trA_1^{2m}B_1^{2m}j + tr\ A_2^{2m}B_2^{2m}k$

$\leq (tr\ A_0^{4m}B_0^{4m})^{\frac{1}{2}} + (trA_1^{4m}B_1^{4m})^{\frac{1}{2}}j + (tr\ A_2^{4m}B_2^{4m})^{\frac{1}{2}}k$

$\leq (trA^{4m}B^{4m})^{\frac{1}{2}}\ \leq (trA^{4m})^{\frac{1}{2}}(tr\ B^{4m})^{\frac{1}{2}}$

Therefore, $trA^{2m}B^{2m} \leq\ (trA^{4m})^{\frac{1}{2}}(tr\ B^{4m})^{\frac{1}{2}}$ … (3)

$tr(AB)^{2m} \leq\ tr(A^2B^2)^m \leq trA^{2m}B^{2m} \leq\ (trA^{4m})^{\frac{1}{2}}(tr\ B^{4m})^{\frac{1}{2}}$

The proof of (i) part is completed.

(ii) To prove, $tr(AB)^{2m} \leq \mathrm{tr}(A^2B^2)^m \leq (\mathrm{tr}\ A^2B^2)^m$

$tr(AB)^{2m} = tr(A_0B_0 + A_1B_1j + A_2B_2k)^{2m}$

$\leq tr(A_0^2B_0^2)^m + tr(A_1^2B_1^2)^mj + tr(A_2^2B_2^2)^mk \leq tr(A^{2m}B^{2m}) \leq \mathrm{tr}(A^2B^2)^m$

Therefore, $tr(AB)^{2m} \leq\ tr(A^2B^2)^m$ … (4)

$tr(A^2B^2)^m = tr(A_0^2B_0^2)^m + tr(A_1^2B_1^2)^mj + tr(A_2^2B_2^2)^mk$

$\leq\ (tr\ A_0^2B_0^2)^m + (trA_1^2B_1^2)^mj + (tr\ A_2^2B_2^2)^mk\ \leq\ \mathrm{tr}(A^2B^2)^m$

Therefore, $\mathrm{tr}(A^2B^2)^m \leq \mathrm{tr}(A^2B^2)^m$ … (5)

Combining (4) and (5), we get,

$tr(AB)^{2m} \leq\ tr(A^2B^2)^m \leq (tr\ A^2B^2)^m$

The proof of (ii) part is completed.

(iii) From (i) part we get,

$tr(AB)^{2m} \leq\ tr(A^2B^2)^m \leq tr\ A^{2m}B^{2m}$ … (6)

Now, $trA^{2m}B^{2m} = tr\ (A_0^{2m}B_0^{2m} + A_1^{2m}B_1^{2m}j + A_2^{2m}B_2^{2m}k)$

$\leq tr(A_0^2 + A_1^2j + A_2^2k)tr(A_0^{2(m-1)} + A_1^{2(m-1)}j + A_2^{2(m-1)}k)tr(B_0^{2m} + B_1^{2m}j + B_2^{2m}k)$

$\leq tr\ A^2\ trA^{2(m-1)}trB^{2m}$

Therefore, $trA^{2m}B^{2m} \leq tr\ A^2\ trA^{2(m-1)}trB^{2m}$ … (7)

Now, $tr\ A^2\ trA^{2(m-1)}trB^{2m} \leq (tr(A_0^2 + A_1^2 j + A_2^2 k)^{\frac{1}{2}})^2 (tr(A_0^2 + A_1^2 j + A_2^2 k))^{m-1}$

$(tr(B_0^2 + B_0^2 j + B_2^2 k))^m$, when $m \geq 1$

$\leq \left(tr(A^2)^{\frac{1}{2}}\right)^2 (trA^2)^{m-1}(trB^2)^m$, when $m \geq 1$

Therefore, $trA^2 trA^{2(m-1)}trB^{2m} \leq \left(tr(A^2)^{\frac{1}{2}}\right)^2 (trA^2)^{m-1}(trB^2)^m$ … (8)

Combining (6), (7) and (8), we get

$tr(AB)^{2m} \leq\ tr(A^2B^2)^m \leq tr\ A^{2m}B^{2m} \leq trA^2 trA^{2(m-1)}trB^{2m}$

$\leq \left(tr(A^2)^{\frac{1}{2}}\right)^2 (trA^2)^{m-1}(trB^2)^m$

The proof is completed.

Corollary 3.4:

Let $A, B \in H_0^+(n), m \in \mathbb{N}$, then the inequalities

$tr(AB)^m \leq (trA^{2m})^{\frac{1}{2}}(trB^{2m})^{\frac{1}{2}}, m \in N$ and $tr(AB)^m \leq (tr\ AB)^m$, $m \in N$ hold. Moreover when $m \geq 1$, it follows that

$0 \leq tr(AB)^{2m} \leq tr\ (A^2B^2)^m \leq tr\ A^{2m}\ B^{2m} \leq tr\ A^2\ trA^{2(m-1)}trB^{2m}$

$\leq (trA)^2\ (trA^2)^{m-1}(trB^2)^m$ … (3.7)

$0 \leq tr(AB)^{2m+1} \leq tr\ A^{2m+1}B^{2m+1} \leq trA\ trBtrA^{2m}trB^{2m} \leq trA\ trB\ (trA^2)^m(trB^2)^m$

… (3.8)

Proof:

Using $A^{\frac{1}{2}}, B^{\frac{1}{2}}$ instread of A, Bin the inequalities (3.4) and (3.5),

$tr(AB)^m \leq (trA^{2m})^{\frac{1}{2}}(trB^{2m})^{\frac{1}{2}}$ and $tr(AB)^m \leq (trAB)^m, m \in N$ can be obtained.

From $A \in H_0^+(n)$ it follows that $(A^2)^{\frac{1}{2}} = A$ and (3.7) is derived by (3.6).

Further more through (3.6), (3.7), Lemma (2.3) it holds that

$0 \leq tr(AB)^{2m+1}$

$\leq tr\ (A_0B_0 + A_1B_1 j + A_2B_2 k)^{2m+1}$

$= tr((A_0^{\frac{1}{2}} + A_1^{\frac{1}{2}} j + A_2^{\frac{1}{2}} k)^2 (B_0^{\frac{1}{2}} + B_1^{\frac{1}{2}} j + B_2^{\frac{1}{2}} k)^2)^{2m+1}$

$\leq\ tr[(A_0^{\frac{1}{2}} + A_1^{\frac{1}{2}} j + A_2^{\frac{1}{2}} k)^{2(2m+1)} (B_0^{\frac{1}{2}} + B_1^{\frac{1}{2}} j + B_2^{\frac{1}{2}} k)^{2(2m+1)}]$

$= tr[(A_0^{2m+1} + A_1^{2m+1} j + A_2^{2m+1} k)(B_0^{2m+1} + B_0^{2m+1} j + B_2^{2m+1} k)]$

$= tr\ A^{2m+1}B^{2m+1}$

$\leq\ tr\ A^{2m+1}\ tr\ B^{2m+1}$

$$= tr\{(A_0 + A_1j + A_2k)(A_0^{2m} + A_1^{2m}j + A_2^{2m}k)\}tr\{(B_0 + B_1j + B_2k)(B_0^{2m} + B_1^{2m}j + B_2^{2m}k)\}$$

$$= trA\, A^{2m}tr\, B\, B^{2m}$$

$$\leq trA\, tr\, A^{2m}trB\, tr\, B^{2m}$$

$$\leq tr\,(A_0 + A_1j + A_2k)\, tr(A_0^{2m} + A_1^{2m}j + A_2^{2m}k)\, tr(B_0 + B_1j + B_2k)\, tr(B_0^{2m} + B_1^{2m}j + B_2^{2m}k)$$

$$\leq tr\,(A_0 + A_1j + A_2k)tr(B_0 + B_1j + B_2k)tr(A_0^{2m} + A_1^{2m}j + A_2^{2m}k)\, tr(B_0^{2m} + B_1^{2m}j + B_2^{2m}k)$$

$$= tr\, A\, trB\, tr\, A^{2m}tr\, B^{2m}$$

$$\leq tr\,(A_0 + A_1j + A_2k)tr(B_0 + B_1j + B_2k)tr\,(A_0 + A_1j + A_2k)^{2m}tr(B_0 + B_1j + B_2k)^{2m}$$

$$\leq tr\,(A_0 + A_1j + A_2k)tr(B_0 + B_1j + B_2k)tr(A_0^2 + A_1^2j + A_2^2k)^m tr(B_0^2 + B_1^2j + B_2^2k)^m$$

$$\leq trAtrB(trA^2)^m(trB^2)^m$$

giving (3.8), that is,

$$0 \leq tr(AB)^{2m+1} \leq tr\, A^{2m+1}B^{2m+1} \leq trA\, trBtrA^{2m}trB^{2m} \leq trA\, trB\,(trA^2)^m(trB^2)^m$$

The proof is completed.

4. Trace of the Power on Quaternion Herimtian Matrix and Quaternion Skew Hermitian Matrix

Theorem 4.1:

Let $A \in H(n), B \in HS(n)$, then when $m = 4t$ or $m = 4t + 2, t \in \mathbb{N}, tr(AB)^m$ and trA^mB^m are all real numbers and

$$tr(AB)^m \leq tr(A^2B^2)^{\frac{m}{2}} \leq trA^mB^m, m = 4t, t \in \mathbb{N} \qquad \text{…. (4.1)}$$

$$tr(AB)^m \geq tr(A^2B^2)^{\frac{m}{2}} \geq trA^mB^m, m = 4t + 2, t \in \mathbb{N}$$

similarly when $m = 4t + 1$ or $m = 4t + 3, t \in N$, if $tr(AB)^m \neq 0$ or $trA^mB^m \neq 0$ then $tr(AB)^m \notin \mathbb{R}$ or $trA^mB^m \notin \mathbb{R}$, so $tr(AB)^m$ and trA^mB^m cannot be compared with each other.

Proof:

By Lemma (2.5), we have that both $tr(AB)^m$ and trA^mB^m are real numbers when $m = 4t$. Further more though Theorem (3.2) and Lemma (2.5), it follows that,

$$tr(AB)^m = \left(-\sqrt{-1}\right)^{4t} tr\left(A_0 + A_1j + A_2k\left(\sqrt{-1}B_0 + B_1j + B_2k\right)\right)^{4t}$$

$$= tr\left(-\sqrt{-1}\right)^{4t} tr\left(A\left(\sqrt{-1}B\right)\right)^{4t} = tr\left(A_0 + A_1j + A_2k\left(\sqrt{-1}B_0 + B_1j + B_2k\right)\right)^{4t}$$

$$= tr\left(A\left(\sqrt{-1}B\right)\right)^{4t} \leq tr\left(A_0^2 + A_1^2j + A_2^2k\left(\sqrt{-1}B_0 + B_1j + B_2k\right)^2\right)^{2t}$$

$$\leq tr\left(A^2\left(\sqrt{-1}B\right)^2\right)^{2t} = \sqrt{-1}^{4t} tr(A_0^2B_0^2 + A_1^2B_1^2j + A_2^2B_2^2k)^{2t}$$

$$= \sqrt{-1}^{4t} tr(A^2B^2)^{2t} = tr(A_0^2B_0^2 + A_1^2B_1^2j + A_2^2B_2^2k)^{\frac{m}{2}} = tr(A^2B^2)^{\frac{m}{2}}$$

$$\leq tr\, A_0^{4t} + A_1^{4t}j + A_2^{4t}k\left(\sqrt{-1}\, B_0 + B_1j + B_2k\right)^{4t}$$

$$\leq trA\left(\sqrt{-1}B\right)^{4t} = \sqrt{-1}^{4t} tr\{(A_0^{4t} + A_1^{4t}j + A_2^{4t}k)(B_0^{4t} + B_1^{4t}j + B_2^{4t}k)\}$$

$$= \sqrt{-1}^{4t} trA^{4t}B^{4t} = tr\{(A_0^m + A_1^m j + A_2^m k)(B_0^m + B_1^m j + B_2^m k)\} = trA^m B^m$$

Giving (4.1), In the same way, when $m = 4t + 2, tr(AB)^m$ and $trA^m B^m$ are all real numbers and it holds that,

$$tr(AB)^m = \left(-\sqrt{-1}\right)^{4t+2} tr\left(A_0 + A_1 j + A_2 k\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)\right)^{4t+2}$$

$$= \left(-\sqrt{-1}\right)^{4t+2} tr\left(A\left(\sqrt{-1}B\right)\right)^{4t+2}$$

$$= -tr\left(A_0 + A_1 j + A_2 k\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)\right)^{2(2t+1)}$$

$$= -tr\left(A\left(\sqrt{-1}B\right)\right)^{2(2t+1)}$$

$$\geq -tr\left(A_0^2 + A_1^2 j + A_2^2 k\left(-\sqrt{-1}B_0 + B_1 j + B_2 k\right)^2\right)^{2t+1}$$

$$\geq -tr\left(A^2\left(\sqrt{-1}B\right)^2\right)^{2t+1}$$

$$= -\sqrt{-1}^{4t+2} tr\left(A_0^2 + A_1^2 j + A_2^2 k\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)^2\right)^{2t+1}$$

$$= -\sqrt{-1}^{4t+2} tr\left(A^2\left(\sqrt{-1}B\right)^2\right)^{2t+1}$$

$$= tr(A_0^2 B_0^2 + A_1^2 B_1^2 j + A_2^2 B_2^2 k)^{\frac{m}{2}}$$

$$= tr(A^2 B^2)^{\frac{m}{2}} \quad [Since\ A^2 * B^2 = A_0^2 B_0^2 + A_1^2 B_1^2 j + A_2^2 B_2^2 k = A^2 B^2]$$

$$\geq -trA^{2(2t+1)}\left(\sqrt{-1}B\right)^{2(2t+1)}$$

$$= -\sqrt{-1}^{4t+2} trA^{4t+2}B^{4t+2}$$

$$= trA^m B^m$$

Producing (4.2), when $m = 4t + 1\ or\ m = 4t + 3, t \in N$, the result is obtained by Lemma 2.5.

The proof is completed.

Theorem 4.2:

Let $A, B \in HS(n)$, if $m = 2t, t \in N$, then both $tr(AB)^m$ and $trA^m B^m$ are all real numbers and $tr(AB)^m \leq |tr(AB)^m| \leq tr(A^m B^m)^{\frac{m}{2}} \leq trA^m B^m$ holds. … (4.3)

Proof:

By the Lemma (2.2) and Lemma (2.6), it is known that both $tr(AB)^m$ and $trA^m B^m$ are all real numbers and $tr(AB)^m = tr\left(\left(\sqrt{-1}A_0 + A_1 j + A_2 k\right)\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)\right)^m$

$$= tr\left(\left(\sqrt{-1}A\right)\left(\sqrt{-1}B\right)\right)^m$$

$$trA^m B^m = tr\left(\sqrt{-1}A_0 + A_1 j + A_2 k\right)^m\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)^m = tr\left(\sqrt{-1}A\right)^m\left(\sqrt{-1}B\right)^m$$

More over by (3.3), it follows that

$$tr(AB)^m = tr\left(\left(\sqrt{-1}A_0 + A_1 j + A_2 k\right)\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)\right)^{2t}$$

$$= tr\left(\left(\sqrt{-1}A\right)\left(\sqrt{-1}B\right)\right)^{2t}$$

$$\leq \left|tr\left(\left(\sqrt{-1}A_0 + A_1 j + A_2 k\right)\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)\right)^{2t}\right|$$

$$\leq \left|tr\left(\left(\sqrt{-1}A\right)\left(\sqrt{-1}B\right)\right)^{2t}\right| = |tr(A_0B_0 + A_1B_1 j + A_2B_2 k)^m| = |tr(AB)^m|$$

$$\leq tr\left(\left(\sqrt{-1}A_0 + A_1 j + A_2 k\right)^2\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)^2\right)^t$$

$$\leq tr\left(\left(\sqrt{-1}A\right)^2\left(\sqrt{-1}B\right)^2\right)^t$$

$$= tr\left(\sqrt{-1}^4 A_0^2 B_0^2 + A_1^2 B_1^2 j + A_2^2 B_2^2 k\right)^{\frac{m}{2}}$$

$$= tr\left(\sqrt{-1}^4 A^2 B^2\right)^{\frac{m}{2}}$$

$$= tr(A_0^2 B_0^2 + A_1^2 B_1^2 j + A_2^2 B_2^2 k)^{\frac{m}{2}} = tr(A^2 B^2)^{\frac{m}{2}}$$

$$\leq \left(\sqrt{-1}A_0 + A_1 j + A_2 k\right)^{2t}\left(\sqrt{-1}B_0 + B_1 j + B_2 k\right)^{2t}$$

$$\leq tr\left(\sqrt{-1}A\right)^{2t}\left(\sqrt{-1}B\right)^{2t} = tr\left(\sqrt{-1}^{4t}\left(A_0^{2t} B_0^{2t} + A_1^{2t} B_1^{2t} j + A_2^{2t} B_2^{2t} k\right)\right)$$

$$= tr\left(\sqrt{-1}^{4t} A^{2t} B^{2t}\right) = tr(A_0^m B_0^m + A_1^m B_1^m j + A_2^m B_2^m k) = tr A^m B^m \text{ giving (4.3)}$$

The proof is completed.

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