INTRODUCTION AND APPLICATION OF BANACH-STEINHAUS THEOREM IN 2-BANACH SPACES

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ABSTRACT - A White introduced the notion of cauchy sequences in 2-normed spaces. After he also defined 2-Banach spaces during this he introduced the notion of 2-functional and the norm of 2-functional and proved a remarkable theorem MX \[b\] C L X L where L is a 2-Banach space and M and \[b\] are linear manifolds in L [\[b\]] being generated by b which is similar to the Hahn-Banach theorem. And he applied his theorem to obtain some result.

KEYWORD - 2-Banach theorem, 2-normed spaces, etc

INTRODUCTION - Let L be a linear space. The pair \( (L, ||.,||) \) is called a 2-normed space provided
\[ ||a,b|| = 0 \] if and only if a and b are linearly dependent,
\[ ||a,b|| = ||b,a||, \]
\[ ||a,\alpha \beta|| = ||\beta|| ||a,b|| \beta \text{real}, \]
\[ ||a,b+c|| \leq ||a,b|| + ||a,c||. \]
||...|| is called a 2-norm which has been shown in [1] to be non-negative. A linear 2-normed \( (L, ||.,||) \) will simply be denoted by L, unless otherwise stated.

A sequence \( \{x_n\} \) is L is called a Cauchy sequence if there exist \( y, z \in L \) such that y and z are linearly independent and \( \lim_{nm} ||x - x_m,y|| = 0 \), \( \lim_{nm} ||x - x_m,z|| = 0 \). A sequence \( \{x_n\} \) is called convergent if there is an \( x \in L \) such that \( \lim_{nm} ||x - x_m,y|| = 0 \) for all \( y \in L \). In this case we say that \( \{x_n\} \) converges to x, write \( x_n \to x \), and call x the limit of \( \{x_n\} \), A linear 2-normed space in which every Cauchy sequence is convergent is called a 2-Banach space.

A 2-functional is real-valued mapping with domain A × C where A and C are linear manifolds of L. Let F be a 2-functional with domain D(F). F is called a linear 2-functional if:

1. \( F(a + c, b + d) = F(a,b) + F(a,d) + F(c,b) + F(c,d), \)
2. \( F(\alpha a, \beta b) = \beta b F(a,b) \) where \( \alpha, \beta \) are scalars.

Let F be a 2-functional with domain D(F). F is called bounded if there is a constant \( K \geq 0 \) such that \( ||F(a,b)|| \leq K ||a,b|| \) for all \( (a,b) \in D(F) \). If F is bounded then the norm of \( ||F|| \) is given by
\[ ||F|| = \text{glb}\{K: ||F(a,b)|| \leq K ||a,b|| \text{for all}(a,b) \in D(F)\} \]
if F is not bounded, then \( ||F|| = +\infty \)

Clearly the domain of definition of F may in some cases be L × L.

Theorem. Let F be a bounded linear 2-functional with domain D(F). Then
\[ ||F|| = \sup \{|F(x,y)|; ||x,y|| = 1, (x,y) \in D(F)\} \]
We have a theorem on $M \times [b] \subset L \times L$ where $L$ is a 2-Banach space and $M$ and $[b]$ are linear manifolds $L$, $[b]$ being generated by the element $b$, which is similar to the Hahn-Banach theorem.

### Definition:
Let $\{x_n\}$ be an infinite sequence of elements in $L$. The series $\sum_{n=1}^{\infty} x_n$ is said to be convergent in $L$ if the sequence of partial sums $\{S_n\}$ where $S_n = x_1 + x_2 + \ldots + x_n$ is convergent in $L$.

If $S_n \to \infty$ as $n \to \infty$, we write $\sum_{n=1}^{\infty} x_n = S$.

### Definition:
Let $L$ be dimension $\geq 2$ and $a$, $b$ be two linearly independent elements in $L$. Then $L$ is said to have the property (P) with respect to $a$ and $b$ if

$$\|x, a + b\| \leq \lambda \sup_{(x, y) \in D(F)} \|x, y\|$$

for all $x \in L$ and where $\lambda = a$ or $b$.

### Theorem:
Let $L$ be a 2-Banach space of dimension $\geq 2$ and let $a$ and $b$ be two linearly independent elements in $L$. Suppose $L$ has the property (P) with respect to $a$ and $b$. Let $A$ be an index set, be a family of bounded linear 2-functionals with domain $L \times [a + b]$ such that

$$\sup_{i \in A} \|F_i\| < \infty.$$

Then from hypothesis, $M_{n-1}$ is finite. There exists and $x_1 \in L$, $\|x_1,a+b\| = \frac{1}{4}$ and $\|F_1(x_1,a+b)\| > 1$. Suppose in this way it has been possible to select the elements $x_2, x_3, x_4, \ldots, x_{n-1}$ from $L$ and $F_2, F_3, F_4, \ldots, F_{n-1}$ from $\{F_i\}_{i \in A}$ which satisfy (1), for these $n$’s. Let

$$M_{n-1} = \sup_{i \in A} \|F_i(x_1 + x_2 + \ldots + x_{n-1}, a + b)\|.$$
Let \( x_n = \frac{x^n}{4^n \|x^n, a + b\|} \)

Then \( x_n \in L, \|x_n, a + b\| = \frac{1}{4^n} \) and \( |F_n(x_n, a + b)| > \frac{2}{3} |F_n| \frac{1}{4^n} \)

i.e. \( |F_n| < \frac{3}{2} \cdot 4^n |F_n(x_n, a + b)| \).

Further \( |F_n(x_n, a + b)| > \frac{2}{3} \cdot \frac{4^n}{4^n} [M_{n-1} + n] \), by (2)

\( = 2[M_{n-1} + n] \).

We form the infinite series

\( x_1 + x_2 + x_3 + \ldots + x_n + \ldots \)

Which we show first to be convergent Let \( S_n = x_1 + x_2 + \ldots + x_n, n = 1, 2, \ldots \) and \( n > m \). Using the property (P) we see that

\[ \|S_n - S_m, a\| \leq \sum_{i=m+1}^{n} \|x_i, a\| \leq \sum_{i=m+1}^{n} \|x_i, (a + b)\| \]

and similarly

\[ \|S_n - S_m, b\| \leq \sum_{i=m+1}^{n} \|x_i, (a + b)\| \]

Since \( \|x_n, (a + b)\| = \frac{1}{4^n} \) \( n = 1, 2, \ldots \) the sequence \( \{S_n\} \) is Cauchy. So \( \{S_n\} \) converges to an element \( x \), say of \( L \) i.e. \( \sum_{n=1}^{\infty} x_n = x \in L \)

Now

\[ |F_n(x_{n+1} + x_{n+2} + \ldots, a + b)| \leq |F_n\{x_{n+1}, a + b\} + \|x_{n+2}, a + b\| + \ldots\} \]

\[ = |F_n\{\frac{1}{4^{n+1}} + \frac{1}{4^{n+2}} + \ldots\} \]

\[ = |F_n\| \frac{1}{4^n} \cdot \frac{1}{3}. \]

(5)

From (3) and (5) we obtain

\( |F_n(x_{n+1} + x_{n+2} + \ldots, a + b)| \leq \frac{1}{2} F_n(x, a + b). \)

(6)

Now, \( |F_n(x, a + b)| = |F_n(x_1 + x_2 + \ldots + x_n + x_{n+1} + \ldots, a + b)| \)

\[ = |F_n\{x_n + (x_1 + x_2 + \ldots + x_{n-1}) + x_{n+1} + \ldots, a + b\} | \]

\[ \geq |F_n(x_n, a + b)| - |F_n(x_1 + x_2 + \ldots, a + b)| \]

\[ - |F_n(x_{n+1} + x_{n+2} + \ldots, a + b)| \]

\[ \leq |F_n(x, a + b)| - |F_n(x_1 + x_2 + \ldots, a + b)| \]

\[ - |F_n(x_{n+1} + x_{n+2} + \ldots, a + b)| \]
\[|F_n(x,a+b)| - \frac{1}{2} |F_n(x_n,a+b)| \geq |F_n(x_1+x_2+\ldots+x_{n-1},a+b)|, \text{ by (6)}\]

\[= \frac{1}{2} |F_n(x_n,a+b)| - |F_n(x_1+x_2+\ldots+x_{n-1},a+b)|.\]

Using (4) and the definition of $M_{n-1}$, it follows that

\[|F_n(x,a+b)| \geq \frac{1}{2} |(x_n,a+b) - M_{n-1}|.\]

\[> 2[M_{n-1} + n] \frac{1}{2} - M_{n-1} = n.\]

This contradiction proves the theorem.

**APPLICATION** Let $L$ denote the set of all polynomials

\[x(t) = x_n + x_1t + x_2t^2 + \ldots + x_nt^n\]

of degree $n$ where $x_i, 0 \leq i \leq n$ are real numbers. Here the positive integer $n$ is not fixed. With the usual definition of addition and scalar (real) multiplication, $L$ is a linear space. We define

\[\|x, y\| = 0 \text{ if } x \text{ and } y \text{ are linearly dependent and}\]

\[\|x, y\| = \max \{|x_i| \cdot \max |x_j|\}, \quad i, j,
\]

Where $y(t) = y_0 + y_1t + \ldots + y_mt^m \in L$. It may be easily verified that

\[\|x, y\|\]

is a 2-norm on $L$.

Since $i = l$ and $j = t$ are two linearly independent elements of $L$, the dimension of $L$ is at least two. Let $x \in L$ and let

\[x(t) = x_0 + x_1t + \ldots + x_nt^n,\]

Then

\[\|x, i\| = \max \|x_k\| = \|x, j\| = \|x, i+j\|.
\]

This shows that $L$ has the property (P) with respect to $I$ and $j$. We write a polynomial $x(t) \in L$ of degree $N_x$ in the form

\[x(t) = \sum_{j=0}^{N_x} x_j t^j\]

where $x_i = 0$ for $j > N_x$. If $x \in L$ we construct a sequence \{\$F_n\} of 2-functionals on $Lx[1+t]$ by
\( F_n(x, y) = \left(x_0 + x_1 + \ldots + x_{n-1}\right) \lambda \)

Where \( y = \lambda (1 + t) \) and \( \lambda \) is real. Let \( n \) be fixed and \( x, y \in [1 + t] \). Suppose the degree of \( x \) be \( N_x \) and the degree of \( y \) be \( N_y \). Then

\[
x = \sum_{j=0}^{\infty} x_j t^j, x_j = 0 \quad \text{for} \quad j > N_x \quad \text{and} \quad y = \sum_{j=0}^{\infty} y_j t^j, y_j = 0 \quad \text{for} \quad j > N_y.
\]

Then

\[
x + y = \sum_{j=0}^{\infty} (x_j + y_j) t^j
\]

where

\[
x_j + y_j = x_j \quad \text{for} \quad N_x > N_y; N_y < j \leq N_x,
\]

\[
y_j = y_j \quad \text{for} \quad N_y > N_x; N_x < j \leq N_y,
\]

\[
x_j = 0 \quad \text{for} \quad j > \max\{N_x, N_y\}.
\]

Also \( u = \lambda (1 + t) \) and \( v = u (1 + t) \) where \( \lambda \) and \( u \) are real. It is seen easily that

\[
F_n(x, y, u + v) = F_n(x, u) + F_n(x, v) + F_n(y, u) + F_n(y, v)
\]

and

\[
F_n(\alpha x, \beta u) = \alpha \beta F_n(x, u)
\]

where \( \alpha, \beta \) are real numbers. Therefore \( F_n \) is linear for each \( n \).

\[
|F_n(x, u)| = \left|\sum_{j=0}^{n-1} x_j \lambda^j\right| \leq \sum_{j=0}^{n-1} |x_j| \lambda^j \leq n \max |x_j| \leq nx
\]

Also

\[
x \max \|x_j\| = n \|x, u\|, j
\]

so that \( F_n \) is bounded for each \( n \). If \( x \in L \) then \( x(t) \) is a polynomial of degree \( \max |x_j| N_x \) which has at most \( N_x + 1 \) non-zero co-efficients and therefore

\[
|F_n(x, 1 + t)| \leq (N_x + 1) \max |x_j| j
\]

for each \( n \) where \( j \) is taken over \( x_0, x_1, \ldots, x_N \). Therefore the sequence \( \{F_n(x, 1 + t)\} \) is bounded for each \( x \in L \). On the other hand, if \( x(t) = 1 + t + t^2 + t^n \) then \( \|x, u\| = |\lambda| \) and \( F_n(x, u) = (1 + 1 + \ldots + 1) \lambda. \) So \( F_n(x, u) = n |\lambda| = n \|x, u\| \). So,

\[
\|F_n\| = \frac{|F_n(x, u)|}{\|x, u\|} = n
\]

Which shows that \( \{\|F_n\|\} \) is not bounded. Therefore by the above theorem \( L \) with the above 2-norm is not a 2-Banach space.

References —