Product cordial labeling of one point union graphs related to shell graphs

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Abstract. We discuss one point union graphs of shell S_4 related graphs. We discuss $S_4^{(k)}$, $Fl(S_4)^{(k)}$ etc for Product cordial labeling.

Key words: labeling, cordial, product, wheel, crown. tail graph.
Subject Classification: 05C78

2. Introduction: The graphs we consider are simple, finite, undirected and connected. For terminology and definitions we depend on Graph Theory by Harary [8], A dynamic survey of graph labeling by J.Gallian [7] and Douglas West.[8]. I.Cahit introduced the concept of cordial labeling [6].There are variety of cordial labeling available in labeling of graphs. Sundaram, Ponraj, and Somasundaram [9] introduced the notion of product cordial labeling. A product cordial labeling of a graph G with vertex set V is a function f from V to {0, 1} such that if each edge uv is assigned the label f(u)f(v), the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a product cordial graph. We use $v_f(0,1) = (a, b)$ to denote the number of vertices with label 1 are a in number and the number of vertices with label 0 are b in number. Similar notion on edges follows for $e_f(0,1) = (x, y)$.
A lot of work is done in this type of labeling so far. One interested in survey may refer Dynamic survey in Graph labeling by J. Gallian.We mention a very short part of it. Sundaram, Ponraj, and Somasundaram have shown that trees; unicyclic graphs of odd order; triangular snakes; dragons; helms; Pm∪Pn; Cm∪Pn; Pm∪K1,n; Wm∪Fn (Fn is the fan Pn+K1); K1,m∪K1,n; Wm∪ K1,n; Wm ∪Pn; Wm ∪Cn; the total graph of P_n (the total graph of P_n has vertex set V (Pn)∪E(Pn) with two vertices adjacent whenever they are neighbors in Pn); Cn if and only if n is odd; $C_n^{(t)}$, the one-point union of t copies of C_n, provided t is even or both t and n are even; K2+mK1 if and only if m is odd; $C_m \cup P_n$ if and only if m+n is odd; $K_{m,n} \cup Ps$ if s > mn; Cn+2∪K1,n; Kn∪Kn,(n−1)/2 when n is odd; Kn∪Kn−1,n/2 when n is even; and P2 n if and only if n is odd. They also prove that $K_{m,n}$ (m,n > 2), $P_m \times P_n$ (m,n > 2) and wheels are not product cordial and if a (p,q)-graph is product cordial graph, then q 6 (p−1)(p + 1)/4 + 1. In this paper We show that one point union of G = FL(C_3), , bull of C_3, crown of C_3, double crown of C_3, C_3^{++}, tail(C_3,$2P_2$), C_3 attached with 2 pendent edges attached at adjacent vertices and show them to be Product cordial under certain conditions.

3. Preliminaries:

3.1 Fusion of vertex. Let G be a (p, q) graph. Let u≠v be two vertices of G. We replace them with single vertex w and all edges incident with u and that with v are made incident with w. If a loop is formed is deleted. The new graph has p-1vertices and at least q-1 edges. If u∈G_1 and v∈G_2, where G_1 is (p_1,q_1) and G_2 is (p_2,q_2) graph. Take a new vertex w and all the edges incident to u and v are joined to w and vertices u and v are deleted. The new graph has p_1+p_2-1 vertices and $q_1 + q_2$ edges. Sometimes this is referred as "u is identified with ..".The concept is well elaborated in John Clark and D. Holton [6]

3.2 Crown graph. It is C_n □K_2.At each vertex of cycle a n edge was attached. We develop the concept further to obtain crown for any graph. Thus crown (G) is a graph G □K_2.It has a pendent edge attached to each of it's vertex. If G is a (p,q) graph then crown(G) has q+p edges and 2p vertices. 3.3 Flag of a graph G denoted by FL(G) is obtained by taking a graph G=G(p,q).At suitable vertex of G attach a pendent edge.It has p+1 vertices and q+1 edges.

3.4 $G^{(K)}$ it is One point union of k copies of G is obtained by taking k copies of G and fusing a fixed vertex of each copy with same fixed vertex of other copies to create a single vertex common to all copies. If G is a (p, q) graph then $|V(G_{(k)}| = k(p-1)+1$and $|E(G)|= k.q$

3.5 A bull graph bull(G) was initially defined for a C_3-bull.It has a copy of G with an pendent edge each fused with any two adjacent vertices of G. For G is a (p,q) graph, bull(G) has p+2 vertices and q+2 edges.

3.6 A tail graph (also called as antenna graph) is obtained by fusing a path p_k to some vertex of G. This is denoted by tail(G, P_k). If there are t number of tails of equal length say (k-1) then it is denoted by tail(G, tp_k). If G is a (p,q) graph and a tail P_k is attached to it then tail(G, P_k) has p+k-1 vertices and q+k-1 edges.

4. Main Results:

Theorem 4.1 Let G = $S_4^{(K)}$.Then G is product cordial iff k is an even number.

Proof: We define i^{th} copy of S_4 in G as : the cycle C_4 of S_4 as ($u_{i,1}$,$c_{i,1}$,$u_{i,2}$,$c_{i,2}$,$u_{i,3}$,$c_{i,3}$,$u_{i,4}$,$c_{i,4}$,$u_{i,1}$); the chord ($u_{i,1}u_{i,3}$) ; i = 1, 2, ..k.

Define a function f: V(G) → { 0,1} as follows:
Case k = 2x, an even number.
The vertex common to all copies of S_4 is say $u_{i,1}$.
$f(u_{i,j})$ = 1 for all i = 1, 2, ..x, and all j = 1, 2, 3, 4.
$f(u_{i,j})$ = 0 for all i = x+1,.. 2x, and all j = 2, 3, 4.

The label number distribution is $v_f(0,1)= (3x,3x+1)$; $e_f(0,1) =(5x,5x)$.If we change the common point on G the same function f works as product cordial labeling. The only one thing is required is that the vertex common to all copies of S_4 is labeled as '1'.

Case K is an odd number say 2x+1.

We need $v_f(0) = v_f(1)$.This produces $e_f(0) = e_f(1)+2$ at least. Therefore the condition $| e_f(0)- e_f(1)|\leq 1$ is not satisfied. The graph is not product cordial.

Theorem 4.2 Let $G' = FL(S_4)$ then $G = G'^{(k)}$ is product cordial for all k if one point union is taken at any vertex other than pendent vertex. If the common vertex is taken at pendent vertex then G is product cordial iff k is even.

Proof: We define i^{th} copy of $FL(S_4)$ in G as : the cycle C_4 of S_4 as $(u_{i,1},c_{i,1},u_{i,2},c_{i,2},u_{i,3},c_{i,3},u_{i,4},c_{i,4},u_{i,1})$; the chord $(u_{i,1}u_{i,3})$; i = 1, 2, ..k. and the pendent vertex $u_{i,5}$ with corresponding pendent edge $(u_{i,1}u_{i,5})$.Thus G has 4k+1 vertices and 6k edges. In structure 1 we take vertex common to all copies as $u_{i,1}$, the degree three vertex on S_4. In structure 2 we take vertex common to all copies as $u_{i,2}$, the degree two vertex on S_4.

Define a function f: V(G) → { 0,1} as follows: (The one point union of $FL(S_4)$ is taken at vertex S4.

Case k = 2x, an even number.

Let the common point to all copies of $FL(S_4)$ be $u_{i,1}$

$f(u_{i,j}) = 1$ for all i = 1, 2, ..x, and all j = 1, 2, 3, 4, 5.

$f(u_{i,j}) = 0$ for all i = x+1,.. 2x, and all j = 2, 3, 4, 5.

The label number distribution is $v_f(0,1)= (4x, 4x+1)$; $e_f(0,1) =(6x,6x)$.

If we change the common point on G the same function f works as product cordial labeling. The only one thing is required is that the vertex common to all copies of $FL(S_4)$ is labeled as '1'.

Case k = 2x+1. Let the common point to all copies of S_4 be $u_{i,1}$

$f(u_{i,j}) = 1$ for all i = 1, 2, ..x, and all j = 1, 2, 3, 4, 5.

$f(u_{i,j}) = 0$ for all i = x+1,.. 2x, and all j = 2, 3, 4, 5.

$f(u_{i,j}) = 1$ for i = 2x+1, and all j = 1, 2, 3;

$f(u_{i,j}) = 1$ for i = 2x+1, and all j = 4, 5.

The label number distribution is $v_f(0,1)= (4x+2,4x+3)$; $e_f(0,1) =(6x+3,6x+3)$.If we change the common point on G the same function f works as product cordial labeling. The only one thing is required is that the vertex common to all copies of $FL(S_4)$ be labeled as 1. (in all above cases it was $u_{i,1}$)

If the one point union of $FL(S_4)$ is taken at pendent vertex of $FL(S_4)$ then for k = 2x only the G is product cordial graph. The same function as above will work with condition that the vertex common to all copies of $FL(S_4)$ be labeled as 1.When k = 2x+1 this graph is not product cordial.

Theorem 4.3 Let $G' = Bull(S_4)$.Then $G= G'^{(k)}$ is product cordial for all k when common vertex is a cycle C_4 vertex and if the common point is a pendent vertex then for k = 2x only.

Proof: We define i^{th} copy of $Bull(S_4)$ in G as : the cycle C_4 of S_4 as $(u_{i,1},c_{i,1},u_{i,2},c_{i,2},u_{i,3},c_{i,3},u_{i,4},c_{i,4},u_{i,1})$; the chord $(u_{i,1}u_{i,3})$, two pendent edges as $(u_{i,1}u_{i,5})$, $(u_{i,2}u_{i,6})$ corresponding to pendent vertices $u_{i,5}$, $u_{i,6}$; i = 1, 2, ..k. Thus G has 5k+1 vertices and 7k edges.

Case k = 2x, an even number.

Let the common point to all copies of $bull(S_4)$ be $u_{i,1}$

$f(u_{i,j}) = 1$ for all i = 1, 2, ..x, and all j = 1, 2, .., 6.

$f(u_{i,j}) = 0$ for all i = x+1,.. 2x, and all j = 2, 3, …,6

The label number distribution is $v_f(0,1)= (5x, 5x+1)$; $e_f(0,1) =(7x,7x)$.If we change the common point on G to some other vertex on cycleC_4 of S_4 (i.e the vertices $u_{i,2}$, $u_{i,3}$,$u_{i,4}$)same function f works as product cordial labeling.

Case k = 2x+ 1, an odd number.

Let the common point to all copies of $bull(S_4)$ be $u_{i,1}$

$f(u_{i,j}) = 1$ for all i = 1, 2, ..x, and all j = 1, 2, .., 6.

$f(u_{i,j}) = 0$ for all i = x+1,.. 2x, and all j = 2, 3, …,6

$f(u_{i,j}) = 1$ for all i = 2x+ 1 and all j = 1, 2, 3.

$f(u_{i,j}) = 0$ for all i = x+1,.. 2x, and all j = 4, 5, 6.

The label number distribution is $v_f(0,1)= (4x+3, 4x+3)$; $e_f(0,1) =(7x+4,7x+3)$.

If we change the common point on G to some other vertex on cycle C_4 of S_4 (i.e the vertices $u_{i,2}$, $u_{i,3}$,$u_{i,4}$) same function f works as product cordial labeling.

If the common vertex to all k copies of $bull(S_4)$ in G is taken as a pendent vertex ($u_{i,5}$ or $u_{i,6}$) then the same function as above gives us product cordial labeling for k = 2x an even number. Also the label number distribution is same. But when k = 2x+1 the graph is not product cordial.

Theorem 4.4 The one point union of k copies of S_4^+ i.e. $G= (S_4^+)^{(k)}$ is product cordial.

Proof: We define i^{th} copy of S_4^+ in G as : the cycle C_4 of S_4 as $(u_{i,1},c_{i,1},u_{i,2},c_{i,2},u_{i,3},c_{i,3},u_{i,4},c_{i,4},u_{i,1})$; the chord $(u_{i,1}u_{i,3})$, four pendent edges as $(u_{i,j}u_{i,j+4})$; j= 1,2, 3, 4. Corresponding to pendent vertices $u_{i,j}$,i = 1, 2, ..k and j = 5,6,7,8. Thus G has 7k+1 vertices and 9k edges.

Case k = 2x, an even number.

Let the common point to all copies of G be $u_{i,1}$ (or any one on S_4)

$f(u_{i,j}) = 1$ for all i = 1, 2, ..x, and all j = 1, 2, .., 8.

$f(u_{i,j}) = 0$ for all i = x+1,.. 2x, and all j = 2, 3, …,8

The label number distribution is $v_f(0,1)= (7x, 7x+1)$; $e_f(0,1) =(9x,9x)$.

Case k = 2x+1, an even number.

Let the common point to all copies of G be $u_{i,1}$

$f(u_{i,j}) = 1$ for all i = 1, 2, ..x, and all j = 1, 2, .., 8.
$f(u_{i,j}) = 0$ for all i = x+1,.. 2x, and all j = 2, 3, …,8.
$f(u_{i,j}) = 1$ for all i = 1, 2, ..x, and all j = 1, 2,3,5
$f(u_{i,j}) = 0$ for all i = x+1,.. 2x, and all j = 4,6,7,8
The label number distribution is $v_f(0,1)$= (7x+3, 7x+4) ; $e_f(0,1)$ =(9x+5,9x+4).
Let the common point be a pendent vertex say $u_{i,5}$.
Labels up to first 2x copies are same as above. Then labels on $(2x+1)^{th}$ copy of S_4^+ are given by : $f(u_{i,j}) = 1$ for all i = 2x+1, and all j = 1, 2,3,5.
$f(u_{i,j}) = 0$ for all i = 2x+1, and all j = 4,6,7,8.
The label number distribution is $v_f(0,1)$ = (7x+3, 7x+4) ; $e_f(0,1)$ = (9x+5, 9x+4).

Theorem 4.5 Let G' be the graph obtained by fusing two pendent edges at any vertex of S_4. G' = (tail(S_4,2P_2).Then $G'^{(k)}$ is product cordial for all k if the common vertex is any vertex which is not pendent vertex. If common vertex is pendent vertex then k = 2x only.

Proof: We define i^{th} copy of G' in G as : the cycle C_4 of S_4 as ($u_{i,1}$,$c_{i,1}$,$u_{i,2}$,$c_{i,2}$,$u_{i,3}$,$c_{i,3}$,$u_{i,4}$,$c_{i,4}$,$u_{i,1}$); the chord ($u_{i,1}u_{i,3}$), two pendent edges as ($u_{i,1}u_{i,5}$), ($u_{i,1}u_{i,6}$). Corresponding to pendent vertices $u_{i,j}$,i = 1, 2, ..k and j = 5, 6. Thus G has 5k+1 vertices and 7k edges. There are two structures. In structure 1 we take the common point as a point on S_4 in G. In structure 2 we take pendent vertex as a common point on G.

Structure 1:
Case k = 2x, an even number.
$f(u_{i,j}) = 1$ for all i = 1, 2, ..x, and all j = 1, 2, .., 6.
$f(u_{i,j}) = 0$ for all i = x+1,.. 2x, and all j = 2, 3, …,6.
The label number distribution is $v_f(0,1)$= (5x, 5x+1) ; $e_f(0,1)$ =(7x,7x).
Case k = 2x+1, an odd number.
$f(u_{i,j}) = 1$ for all i = 1, 2, ..x, and all j = 1, 2, .., 6.
$f(u_{i,j}) = 0$ for all i = x+1,.. 2x, and all j = 2, 3, …,6.
$f(u_{i,j}) = 1$ for all i = 2x+1. and all j = 1, 2, 3.
$f(u_{i,j}) = 0$ for all i = 2x+1. and all j = 4, 5, 6.
The label number distribution is $v_f(0,1)$= (5x+2, 5x+3) ; $e_f(0,1)$ =(7x+4,7x+3).

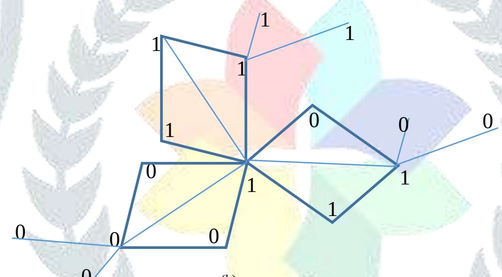

Fig 4.2 $G'^{(k)}$ $v_f(0,1)$ = (8,8); $e_f(0,1)$=(11,10)

Structure 2:
Case k = 2x, an even number.
$f(u_{i,j}) = 1$ for all i = 1, 2, ..x, and all j = 1, 2, .., 6.
$f(u_{i,j}) = 0$ for all i = x+1,.. 2x, and all j = 2, 3, …,6.
The label number distribution is $v_f(0,1)$= (5x, 5x+1) ; $e_f(0,1)$ =(7x,7x).
Case k = 2x+1, an odd number. Then there is no product cordial labeling.

Conclusions: In this paper we have obtained one point union graphs on S_4,and and the graphs obtained from S_4 by fusing pendent edges (maximum two at a vertex) at vertices of S_4.We have studied these path unions for product cordiality. We have shown that:

1) $S_4^{(K)}$ is product cordial iff k is an even number.

2) Let G' = FL(S_4) then G = $G'^{(k)}$ is product cordial for all k if one point union is taken at any vertex other than pendent vertex. If the common vertex is taken at pendent vertex then G is product cordial iff k is even.

3) Let G' = Bull(S_4).Then G= $G'^{(k)}$ is product cordial for all k when common vertex is a cycle C_4 vertex and if the common point is a pendent vertex then for k = 2x only.

4)The one point union of k copies of S_4^+ i.e. G= $(S_4^+)^{(k)}$ is product cordial.

5) Let G' be the graph obtained by fusing two pendent edges at any vertex of S_4. G' = (tail(S_4,2P_2). Then $G'^{(k)}$ is product cordial for all k if the common vertex is any vertex which is not pendent vertex. If common vertex is pendent vertex then k = 2x only.It is necessary to study these types og graphs further.

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