An Unbroken Pseudo-Bosonic System Under the Second Derivative and Polynomial Super Symmetry

Abhijit Banerjee [1] and Partha Mandal [2]

[1] Department of Mathematics, Krishnath College, Berhampore, Murshidabad, India-742101
[2]Corresponding author: Department of Mathematics, Triveni Devi Bhalotia College, Raniganj, Paschim Bardhaman, India-713347

We have demonstrated a simple theoretical model, involving the pseudo bosons within the second derivative supersymmetric (SSUSY) framework, which under certain suitable constraint produces a triplet of super partner Hamiltonians, that depict polynomial supersymmetry (PSUSY) for pseudo bosonic systems. A class of non singular oscillators have found, as a pseudo bosonic system using SSUSY and PSUSY representations of our choice. From the zero mode states of their corresponding quasi Hamiltonian, we have calculated the Witten index that reveals, the underlying supersymmetry (SUSY) is unbroken in nature.

PACS: 03.65.Fd, 03.65.Ge, 03.65.ca.
Keywords:Supersymmetry, Pseudo boson, SSUSY, PSUSY.

1 Introduction

Over the past few years, research in non-Hermitian quantum mechanics has opened up new directions of enquiry. In this regard, different existing formulations have been reanalyzed and certain aspects actively developed in the area of the system of pseudo bosons. The concept of pseudo bosons arise from the modification of canonical commutation relation for the operators acting in a bosonic system

$$[a, a^{\dagger}] = 1 \tag{1.1}$$

by replacing the commutation relation

$$[a, b] = 1 \tag{1.2}$$

where b is not the adjoint of a.

The pseudo bosons are developed in concrete physical models along the years (see the review [1] and references therein). It appeared in the context of pseudo-hermitian quantum mechanics and particularly in $\mathcal{PT}$-symmetric cases (for detail see the reviews [2, 3]).

In a recent paper [4] Bagarello discussed a general strategy to construct the pseudo-bosonic systems by extending ordinary SUSY and studied how these are related to pseudo-hermitian quantum mechanics. In his work, he constructed a class of shifted oscillators as a pseudo bosonic system. Motivated by this work, in the following, we study the pseudo bosonic system in the perspectives of second derivative SUSY (SSUSY) formalism [5]-[15] involving representations of Darboux factorization operators.

SSUSY schemes have many profound applications in the literature. To name a few significant works, such as coupled channel problems, transparent matrix potential. The SSUSY realisation is controlled by two conserved 2×2 matrix superalgebras $Q^{\pm}$ and a quasi Hamiltonian

$$K = \{Q^{+}, Q^{-}\},$$

which is a fourth order differential operator and also the second order of Schrödinger operator, lead us to the polynomial SUSY (PSUSY) [11, 12, 13] picture.

The organisation of this paper is as follows: in section 2 and 3 we discuss the SSUSY and PSUSY representations of pseudo bosonic system and illustrating them we construct a class of non singular oscillators; in section 4 we study the zero states and Witten index for our model and finally in section 5 we present the summary of our work.

2 Second Derivative Super Symmetry (SSUSY)

For second order derivative generalization of supersymmetric quantum mechanics we consider the most general form of supercharges (basically those involving Darboux type factorization) [5, 11]

$$\begin{aligned}\mathcal{A}^{\pm} &= \frac{1}{2}\partial^2 \pm \{f(x), \partial\} + \phi(x) \\ &= \frac{1}{2}\partial^2 \pm 2f(x)\,\partial + \phi(x) \pm f'(x) \end{aligned} \tag{2.1}$$

where $f(x), \phi(x)$ are arbitrary but non-singular real valued functions, $\partial = \frac{d}{dx}$ and $(\mathcal{A}^+)^\dagger = \mathcal{A}^-$.

In terms of $\mathcal{A}^-$ and $\mathcal{A}^+$ let us now introduce the corresponding supercharges Q^+ and Q^- as

$$Q^+ = \begin{pmatrix} 0 & 0 \\ \mathcal{A}^- & 0 \end{pmatrix}, \quad Q^- = \begin{pmatrix} 0 & \mathcal{A}^+ \\ 0 & 0 \end{pmatrix} \tag{2.2}$$

The quasi-Hamiltonian K which is a fourth order differential operator is given by

$$K = \{Q^+, Q^-\} = \begin{pmatrix} \mathcal{A}^+\mathcal{A}^- & 0 \\ 0 & \mathcal{A}^-\mathcal{A}^+ \end{pmatrix}. \tag{2.3}$$

Suppose there exist an another operator H involving Schrödinger like Hamiltonians $h^{(1)}$ and $h^{(2)}$ as

$$H = \begin{pmatrix} h^{(1)} & 0 \\ 0 & h^{(2)} \end{pmatrix} \tag{2.4}$$

where we define $h^{(1)}$ and $h^{(2)}$ in terms of the potential $V^{(1)}$ and $V^{(2)}$,

$$h^{(1,2)} = -\frac{1}{2}\partial^2 + V^{(1,2)} \tag{2.5}$$

such that H commutes with $Q^{\pm}$,

$$[H, Q^{\pm}] = 0. \tag{2.6}$$

From these above equations we obtain the intertwining relations

$$\mathcal{A}^- h^{(1)} = h^{(2)}\mathcal{A}^-, \quad \mathcal{A}^+ h^{(2)} = h^{(1)}\mathcal{A}^+. \tag{2.7}$$

Substituting (2.1) and (2.5) in the first intertwining relation of (2.7) we obtain

$$V^{(1)} - V^{(2)} = 4f' \tag{2.8}$$

$$2f\{V^{(1)} - V^{(2)}\} = V^{(1)\prime} - 2f'' + \phi' \tag{2.9}$$

$$(\phi - f')\{V^{(1)} - V^{(2)}\} = -\frac{1}{2}(\phi'' - f''') - \frac{1}{2}V^{(1)\prime\prime} + 2fV^{(1)\prime} \tag{2.10}$$

while from the second relation of (2.7) we are led to the constraints

$$V^{(1)} - V^{(2)} = 4f' \quad (identicalto\ (2.8))$$

$$2f\{V^{(1)} - V^{(2)}\} = V^{(2)\prime} + 2f'' + \phi' \tag{2.11}$$

$$(\phi + f')\{V^{(1)} - V^{(2)}\} = \frac{1}{2}(\phi'' + f''') + \frac{1}{2}V^{(2)\prime\prime} + 2fV^{(2)\prime}. \tag{2.12}$$

Then setting (2.8) into (2.9) and (2.11) we can get respectively

$$V^{(1,2)} = 4f^2 \pm 2f' - \phi + \alpha_{1,2} \tag{2.13}$$

where α_1 and α_2 are constants of integration. As a result

$$V^{(1)} - V^{(2)} = 4f' + (\alpha_1 - \alpha_2).$$

On comparison with (2.8) we get $\alpha_1 - \alpha_2 = 0$ and we take each of constants equal to a common constant α.

Now from the relations (2.13) and (2.10) we acquire a first order differential equation in ϕ as

$$\phi' + \frac{2f'}{f}\phi = 8ff' - \frac{f'''}{4f} \tag{2.14}$$

whose solution is stated by

$$\phi = 2f^2 - \frac{1}{4}\frac{f''}{f} + \frac{1}{8}\left(\frac{f'}{f}\right)^2 + \frac{\beta}{f^2} \tag{2.15}$$

where β is an arbitrary constant. Note that from the relations (2.13) and (2.12) no new result can be obtained.

The corresponding potentials $V^{(1,2)}$ then take the explicit form

$$V^{(1,2)} = 2f^2 \pm 2f' + \frac{1}{4}\frac{f''}{f} - \frac{1}{8}\left(\frac{f'}{f}\right)^2 + \alpha - \frac{\beta}{f^2}. \quad (2.16)$$

Now, we are turning to the quasi-Hamiltonian K which can be assumed as a polynomial in H. This guides us to the so-called "polynomial SUSY".

3 Polynomial Super Symmetry (PSUSY)

To address PSUSY we consider the quasi-Hamiltonian K as polynomial in H. In the present case we take K as quadratic in H. Expressing K as

$$K = H^2 + d.$$

Factorization of K requires d to be a perfect square in the form $d = \pm\frac{c^2}{4}$ according as $d >$ or < 0. However for the reducible algebra $d < 0$, in the following we take $d = -\frac{c^2}{4}$. Then K can be written as

$$K = H^2 - \frac{c^2}{4} = \begin{pmatrix} \left(h^{(1)} + \frac{c}{2}\right)\left(h^{(1)} - \frac{c}{2}\right) & 0 \\ 0 & \left(h^{(2)} - \frac{c}{2}\right)\left(h^{(2)} + \frac{c}{2}\right) \end{pmatrix}. \quad (3.1)$$

To reconcile (3.1) and (2.3), we consider the following factorization of $\mathcal{A}^{\pm}$, which is given in (2.1), as

$$\mathcal{A}^{+} = q_1^{+}q_2^{+}, \qquad \mathcal{A}^{-} = q_2^{-}q_1^{-} \quad (3.2)$$

where the operators $q_1^{+}, q_1^{-}, q_2^{+}, q_2^{-}$ acting in the Hilbert space are of the form

$$q_1^{+} = \frac{1}{\sqrt{2}}(-\partial + W_1), \quad q_1^{-} = \frac{1}{\sqrt{2}}(\partial + W_1)$$
$$q_2^{+} = \frac{1}{\sqrt{2}}(-\partial + W_2), \quad q_2^{-} = \frac{1}{\sqrt{2}}(\partial + W_2) \quad (3.3)$$

with $q_1^{+} = (q_1^{-})^{\dagger}, q_2^{+} = (q_2^{-})^{\dagger}$ and $W_{1,2}$ are real valued potentials. We thus have built up a model of PSUSY which we are going to use on Pseudo-Bosons system.

3.1 Pseudo-Bosons constraint

For the pseudo-bosons system [1] the canonical commutation relations do not hold. As its consequence $q_1^{-} \neq (q_2^{+})^{\dagger}$ and so

$$W_1 \neq W_2. \quad (3.1.4)$$

Furthermore for pseudo-bosons system

$$[q_1^{-}, q_2^{+}] = 1$$
$$\Rightarrow W_1 + W_2 = 2x + \delta; \quad \delta \in \mathbb{R} \quad (3.1.5)$$

Now using the constraint relation (3.1.5) into the factorized form of $\mathcal{A}^{\pm}$ we have

$$\mathcal{A}^{+} = \frac{1}{2}[\partial^2 - (2x + \delta)\,\partial + (W_1W_2 - W_{2'})]$$
$$\mathcal{A}^{-} = \frac{1}{2}[\partial^2 - (2x + \delta)\,\partial + (W_1W_2 + W_{1'})]$$

which on comparison with (2.1) gives

$$f = -\frac{1}{4}(2x + \delta) \quad (3.1.6)$$
$$\phi + f' = \frac{1}{2}(W_1W_2 - W_{2'}) \quad (3.1.7)$$
$$\phi - f' = \frac{1}{2}(W_1W_2 + W_{1'}). \quad (3.1.8)$$

We now set

$$W_1 = 2(W - f), W_2 = -2(W + f) \quad (3.1.9)$$

which on substitution into (3.1.5) satisfy the relation (3.1.6) again. Furthermore adding (3.1.7) and (3.1.8) we get

$$2f^2 - \phi = 2W^2 - W'. \quad (3.1.10)$$

3.2 Triplet of Hamiltonians

However going back to the factorization process again, we choose an another constraint

$$q_2^+ q_2^- - \frac{c}{2} = q_1^- q_1^+ + \frac{c}{2} \tag{3.2.1}$$

and then at once see that

$$\begin{aligned}\mathcal{A}^+\mathcal{A}^- &= (q_1^+ q_2^+)(q_2^- q_1^-) = q_1^+(q_2^+ q_2^-)q_1^- \\ &= (q_1^+ q_1^- + \frac{c}{2} + \frac{c}{2})(q_1^+ q_1^- + \frac{c}{2} - \frac{c}{2})\end{aligned}$$

that suggests the interpretation

$$h^{(1)} = q_1^+ q_1^- + \frac{c}{2}. \tag{3.2.2}$$

In a similar fashion we can express

$$\mathcal{A}^-\mathcal{A}^+ = (q_2^- q_2^+ - \frac{c}{2} - \frac{c}{2})(q_2^- q_2^+ - \frac{c}{2} + \frac{c}{2})$$

and so

$$h^{(2)} = q_2^- q_2^+ - \frac{c}{2}. \tag{3.2.3}$$

Thus on using (3.3), the explicit form of the Hamiltonians can be obtained from (3.2.2) and (3.2.3), which are given by

$$h^{(1,2)} = -\frac{1}{2}\partial^2 + V^{(1,2)} \tag{3.2.4}$$

where

$$V^{(1,2)} = 2(W \mp f)^2 - (W' \mp f') \pm \frac{c}{2}. \tag{3.2.5}$$

As a result

$$V^{(1)} - V^{(2)} = -8Wf + 2f' + c$$

and then comparing with (2.8) we obtain

$$W = \frac{c - 2f'}{8f} = -\frac{c+1}{2}\frac{1}{2x+\delta}. \tag{3.2.6}$$

Correspondingly the partner superpotentials (3.1.9) become

$$W_1 = \frac{c - 2f' - 8f^2}{4f} = -\frac{c+1}{2x+\delta} + (x + \frac{\delta}{2}) \tag{3.2.7}$$

$$W_2 = -\frac{c - 2f' + 8f^2}{4f} = \frac{c+1}{2x+\delta} + (x + \frac{\delta}{2}). \tag{3.2.8}$$

On the otherhand the constraint relation (3.2.1) leads to

$$W_1^2 + W_1{}' + c = W_2^2 - W_2{}' - c$$

which is identically satisfied by the results (3.2.7) and (3.2.8).

Note that there exists an intermediate Hamiltonian h, which is superpartner to both $h^{(1)}$ and $h^{(2)}$, such that

$$h^{(1)} = q_1^+ q_1^- + \frac{c}{2}, \quad h = q_1^- q_1^+ + \frac{c}{2}, \quad h^{(2)} = q_2^+ q_2^- - \frac{c}{2} \tag{3.2.9}$$

and it's explicit form is

$$h = -\frac{1}{2}\partial^2 + \left(W_1^2 + W_1{}' + \frac{c}{2}\right).$$

This triplet $h^{(1)}, h$ and $h^{(2)}$ of Hamiltonians furnishes a model of PSUSY for pseudo-bosonic system.

3.3 Construction of pseudo-bosonic oscillators

First substituting f from (3.1.6) into the equation (2.15) we obtain

$$\phi = \frac{1}{2}(x + \frac{\delta}{2})^2 + \frac{1+32\beta}{2}\frac{1}{(2x+\delta)^2}. \tag{3.3.1}$$

Our choice of non-singular ϕ [5] on the whole real line determines the integration constant $\beta = -\frac{1}{32}$. Consequently $V^{(1,2)}$ take the final form as

$$V^{(1,2)} = \frac{1}{2}(x + \frac{\delta}{2})^2 + \alpha \mp 1$$

which is also non-singular.

Finally, turning to the constraint relation (3.1.10) and substituting f, W and ϕ therein we find $c = \pm 1$.

For $c = -1$ we can get $W = 0$ and $W_1 = W_2$ from (3.2.6), (3.2.7) and (3.2.8). But it is a contradiction to the result (3.1.4), which is required for the pseudo-bosonic system. Thus the only possibility left is $c = 1$.

Now we have to calculate the Witten index that gives us a characteristic of SUSY for the system. So we need to get zero-mode states of the hamiltonian.

4 Zero-mode states

Zero-mode states of the quasi-Hamiltonian K can be obtained from solutions of the equations

$$\mathcal{A}^{\pm}\psi_0^{\pm} = 0$$
$$\Rightarrow [\partial^2 \pm 4f\,\partial + 2(\phi \pm f')]\psi_0^{\pm} = 0$$
$$\Rightarrow \psi_0^{\pm} = \left(2\mu^{\pm}\exp[\mp\tfrac{\delta^2}{8}]\right)\exp[\pm\tfrac{1}{2}(x+\tfrac{\delta}{2})^2]\left\{(x+\tfrac{\delta}{2}) - \tfrac{\nu^{\pm}}{4}\right\} \qquad (4.1)$$

where $\mu^{\pm}, \nu^{\pm}$ are normalization constants.

But $(x+\frac{\delta}{2})^2 \to +\infty$ for $x \to \pm\infty$ and so ψ_0^{+} is inadmissible and correspondingly the number of zero-modes of K is $n_+ = 0$. From the expression for ψ_0^{-} in (4.1) it is evident that the number of zero-modes is $n_- = 1$. As a result the Witten index is $\Delta = n_- - n_+ = 1$ and therefore the SUSY is unbroken [5].

The normalization constants are found from the normalization equation $\int_{-\infty}^{+\infty} |\psi_0^{-}|^2 dx = 1$ and we find that the constants $\mu^{-} and \nu^{-}$ are related by the equation

$$(\mu^{-})^2\exp[\frac{\delta^2}{4}]\left\{1 + \frac{(\nu^{-})^2}{8\sqrt{2}}\right\} = \frac{1}{2\sqrt{\pi}}.$$

5 Conclusion

In this work we have demonstrated a simple second order unbroken SUSY model of the pseudo bosons which under certain suitable constraint follows PSUSY scheme. As a result we obtain a triplet of Hamiltonians which are super partners in nature. In our model we have constructed a class of non singular oscillators as a pseudo-bosonic system. After deriving their zero-mode states we have calculated the corresponding Witten index that characterizes the unbroken behaviour of SUSY. Our method can be extended to investigate higher order pseudo-supersymmetric bosonic models also. In near future we are interested to investigate the higher order problems.

Acknowledgement

The authors thank Prof. B Bagchi, Department of Applied Mathematics, University of Calcutta for his valuable advice.

References

[1] F. Bagarello, *A concise review on pseudo-bosons, pseudo-fermions and their relatives* arXiv:1703.06730[math-ph] 2017.

[2] A. Mostafazadeh, *Int. J. Geom. Methods Mod. Phys.***7**, 1191 2010.

[3] C.M. Bender, *Rep. Progr. Phys.*, **70**, 947 2007.

[4] F. Bagarello, *J.Math.Phys.*, **51**, 53508 2010.

[5] A.A. Andrianov, M.V. Ioffe and V.P. Spiridonov, *Phys.Lett.A*, **174**, 273 1993.

[6] A.A. Andrianov, N.V. Borisov, and M.V. Ioffe *Phys. Lett. A*, **105**, 19 1984.

[7] A.A. Andrianov, N.V. Borisov, M.I. Eides and M.V. Ioffe*Phys. Lett. A***109**, 143 1985.

[8] D. J. Fernandez C., *Int. J. Mod. Phys. A*, **12**, 171 1997.

[9] D. J. Fernandez C. and V. Hussin, *J. Phys. A: Math. Gen.*, **32**, 3603 1999.

[10] F. Cannata and M.V. Ioffe, *Phys. Lett. B*, **278**, 399 1992.

[11] B. Bagchi, *Supersymmetry in Quantum and Classical Mechanics* (Chapman and Hall, Boca Ratton, FL, 2000)

[12] B. Bagchi, S. Mallik, and C. Quesne, *Int.J.Mod.Phys.A*, **17**, 51 2002.

[13] B. Bagchi, A. Banerjee, E. Caliceti, F. Cannata, H. B. Geyer, C. Quesne and M. Znojil *Int.J.Mod.Phys.A*, **20**, 7107 2005.

[14] B.Bagchi, A. Banerjee and P. Mandal, *Int.J.Mod.Phys.A*, **30:9**, 1550037 2015.

[15] P. Mandal and A. Banerjee, *Pramana J.Phys*, **88:1**, 1 2017.