Cycle $C_3$ Related one point union Product cordial graphs

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Abstract: In this paper we discuss one point union graphs obtained from cycle related graphs. We show that $G^{(k)}$ is product cordial where $G = FL(C_n)$, bull of $C_n$, crown of $C_n$, double crown of $C_n$, $C_n^+$, tail($C_3, 2P_2$), $C_3$ attached with 2 pendent edges attached at adjacent vertices and show them to be product cordial under certain conditions.

Key words: labeling, cordial, product, wheel, crown. tail graph.

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Introduction: The graphs we consider are simple, finite, undirected and connected. For terminology and definitions we depend on Graph Theory by Harary [8], A dynamic survey of graph labeling by J. Gallian [7] and Douglas West [8]. I. Cahit introduced the concept of cordial labeling [6]. There are variety of cordial labeling available in labeling of graphs. Sundaram, Ponraj, and Somasundaram [9] introduced the notion of product cordial labeling. A product cordial labeling of a graph $G$ with vertex set $V$ is a function $f$ from $V$ to $\{0, 1\}$ such that if each edge $uv$ is assigned the label $f(u)f(v)$, the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a product cordial graph.

A lot of work is done in this type of labeling so far. One interested in survey may refer Dynamic survey in Graph labeling by J. Gallian. We mention a very short part of it. Sundaram, Ponraj, and Somasundaram have shown that trees; unicyclic graphs of odd order; triangular snakes; dragons; helms; $PmUPn$; $CnUPn$; $PmUK1,n$; $WmUFn$ ($Fn$ is the fan $Pn+K1$); $K1,mUK1,n$; $WmU K1,n$; $Wm U Pn$; $Wm UCn$; the total graph of $Pn$ the total graph of $Pn$ has vertex set $V(Pn)UE(Pn)$ with two vertices adjacent whenever they are neighbors in $Pn$); $Cn$ if and only if $n$ is odd; $C_n^{(0)}$, the one-point union of $t$ copies of $C_n$, provided $t$ is even or both $t$ and $n$ are even; $K2+mK1$ if and only if $m$ is odd; $C_nUPa$ if and only if $m+n$ is odd; $K_{m,n}$ $UPs$ if $s > mn$; $Cn+2UK1,n$; $KnUKn$ ($n−1)/2$ when $n$ is odd; $KnUKn−1,n/2$ when $n$ is even; and $P2$ $n$ if and only if $n$ is odd. They also prove that $K_{m,n}$ ($m, n > 2$), $Fm XPn$ ($m, n > 2$) and wheels are not product cordial and if a $(p, q)$-graph is product cordial graph, then $q = 6(p−1)(p + 1)/4 + 1$. In this paper we show that one point union of $G = FL(C_n)$, bull of $C_n$, crown of $C_n$, double crown of $C_n$, $C_n^+$, tail($C_3, 2P_2$), $C_3$ attached with 2 pendent edges attached at adjacent vertices and show them to be product cordial under certain conditions.

3. Fusion of vertex. Let $G$ be a $(p, q)$ graph. Let $u\neq v$ be two vertices of $G$. We replace them with single vertex $w$ and all edges incident with $u$ and that with $v$ are made incident with $w$. If a loop is formed is deleted. The new graph has $p$-vertices and at least $q$-1 edges. If $ueG1$ and $veG2$, where $G1$ is $(p_1,q_1)$ and $G2$ is $(p_2,q_2)$ graph. Take a new vertex $w$ and all the edges incident to $u$ and $v$ are joined to $w$ and vertices $u$ and $v$ are deleted. The new graph has $p_1+p_2−1$ vertices and $q_1 + q_2$ edges. Sometimes this is referred as “$u$ is identified with”$. The concept is well elaborated in D. West [10].

3.1 Crown graph. It is $C_n OK2$. At each vertex of cycle a n edge was attached. We develop the concept further to obtain crown for any graph. Thus crown $(G)$ is a graph $G OK2$. It has a pendent edge attached to each of it’s vertex. If $G$ is a $(p,q)$ graph then crown $(G)$ has $q+p$ edges and $2p$ vertices.

3.3 Flag of a graph $G$ denoted by $FL(G)$ is obtained by taking a graph $G\rightarrow G(p,q)$. At suitable vertex of $G$ attach a pendent edge. It has $p+1$ vertices and $q+1$ edges.

3.4 $G^{(k)}$ it is one point union of $k$ copies of $G$ is obtained by taking $k$ copies of $G$ and fusing a fixed vertex of each copy with same fixed vertex of other copies to create a single vertex common to all copies. If
G is a (p, q) graph then \(|V(G(k))| = k(p-1)+1|E(G)|= kq \\
3.5 A bull graph \( \text{bull}(G) \) was initially defined for a \( C_3 \)-bull. It has a copy of \( G \) with an pendant edge each fused with any two adjacent vertices of \( G \). For \( G \) is a (p,q) graph, \( \text{bull}(G) \) has \( p+2 \) vertices and \( q+2 \) edges.

3.6 A tail graph (also called as antenna graph) is obtained by fusing a path \( p_k \) to some vertex of \( G \). This is denoted by \( \text{tail}(G, P_k) \). If there are \( t \) number of tails of equal length say \((k-1)\) then it is denoted by \( \text{tail}(G, tp_k) \). If \( G \) is a (p,q) graph and a tail \( P_k \) is attached to it then \( \text{tail}(G, P_k) \) has \( p+k-1 \) vertices and \( q+k-1 \) edges.

Main Results:

4.1 Theorem 1. Let \( G' = \text{FL}(C_3) \). Then \( G = G'(k) \) is product cordial iff \( k \) is congruent to \((2 \mod 4)\).

Proof: There are different structures possible on \( G \) depending on the vertex of \( G' \) (vertex common to all copies) used to obtain \( G \). From fig 4.1 it follows that there are three structures possible on \( G \) by taking one point union on vertex a, b or at c. The point of fusion is 'x'.

In structure 1 type A is used and the vertex common to all graphs is \( x \) whose label is ‘1’. In structure 2 type B is used and the vertex common to all graphs is \( x \) whose label is ‘1’. In structure 3 type B is used and the vertex common to all graphs is \( x \) whose label is ‘1’. In all structures the label number distribution is \( v(0,1) = (3x, 4x+1); e(0,1) = (4x,4x) \) for \( G = (G')^{2x} \).

Theorem 2. Let \( G' = \text{bull}(C_3) \). Then \( G = G'(k) \) is product cordial iff \( k \) is congruent to \((2 \mod 4)\).

Proof: One can see that there are three possible non–isomorphic structures on the graph \( G \). We can take one point union on any of the three vertices as shown in figure 4.5. Define \( f: V(G) \rightarrow (0,1) \) as follows. Using \( f \) we get labeled copies as shown in Type A, Type B, type C, type D. For \( k = 1 \) and \( k = 2 \) figures above explains the matter. When \( k = 2x \) we have to repeatedly fuse respective type of labeling for \( x \) times at
The label number distribution is \( v_1(0,1) = (4x, 4x+1); e_1(0,1) = (5x, 5x+3) \). To obtain the labeled copy of \( G^{(2x+1)} \) first obtain labeled copy of \( G^{(2x)} \). With this fuse type A label at vertex \( d \) on it with vertex \( d \) of \( G^{(2x)} \), (obtained from type C label) fuse type A label at vertex \( y \) on it with vertex \( y \) of \( G^{(2x)} \), (obtained from type D label). Thus we get a labeled copy of \( G^{(2x+1)} \). The label number distribution is \( v_1(0,1) = (4x+3, 4x+3); e_1(0,1) = (5x+5, 5x+5) \). Thus the graph is having product cordial label.

Theorem 4.3 One point union of \( k \) copies of \( C_s \), \( G = (C_s)^{(k)} \) is product cordial graph for all \( k \) provided the point common to all copies is the pendent vertex and common point is degree three then \( k \) is equal to 1 or even number only.

Proof: From figure it follows that on \( C_s^{(k)} \) there are only two structures possible up to isomorphism

Define \( f: V(G) \to \{0,1\} \) as follows. Using \( f \) we get labeled copies of \( G^{(2x)} \) as shown in fig 4.11 and 4.12, fig 4.13. When \( k = 1 \) use type C label. S structure 1. The common point is vertex \( 'b' \). First obtain labeled copy of \( G \) for \( k = 2x \). This is done by fusing type B label at point \( b \) for \( x \) times. The resultant graph is \( C_s^{(2x)} \) and label number distribution is \( v_1(0,1) = (5x, 5x+1); e_1(0,1) = (6x, 6x) \). To obtain a labeled copy for \( k = 2x+1 \) first obtain labeled copy for \( k = 2x \). Append it with type C label at vertex \( b \). The resultant graph is \( C_s^{(2x+1)} \) and label number distribution is \( v_1(0,1) = (5x+3, 5x+3); e_1(0,1) = (6x+3, 6x+3) \).

If \( k \) is odd number greater than 1, if we have to take union point on \( G \) as pendent vertex then if it’s label is ‘0’ then it produces \( e_1(0) \) greater than 2 by \( e_1(1) \). If we label the common vertex as ‘1’ then also condition on edges is not satisfied. When \( k = 2x \) fuse the type A labeling with type A label at point ‘a’ for \( x \) times to get \( G \). The resultant graph is \( C_s^{(2x)} \) and label number distribution is \( v_1(0,1) = (5x, 5x+1); e_1(0,1) = (6x, 6x) \). That completes the proof.

# Theorem 4.4 Let \( G' \) be \( \text{Tail}(C_s, 2P_2) \) obtained from attaching two pendent edges at a vertex of \( C_s \) then \( G = (G')^{(k)} \) is product cordial for all \( k \) and all pairwise non-isomorphic structures obtained by taking different vertices on \( G' \) as common point.

Proof: From fig 4.13 it is clear that we can take one point union at five points but only at three points vertex \( a \), vertex \( b \), vertex \( c \) results in pairwise non-isomorphic structures.
Define \( f: V(G) \rightarrow \{0,1\} \) as follows. Using \( f \) we get labeled copies of \( G'^{(2)} \) as Type A, Type B, Type C and when \( k = 1 \) we have Type D label. To obtain a labeled copy of \( G'^{(k)} \) we first obtain a labeled copy of \( G'^{(2^k)} \). In structure 1 ‘a’ is the common point to all copies of \( G' \). We fuse Type A label repeatedly for \((x-1)\) times to obtain a labeled copy of \( G'^{(2^x)} \). In structure 2 ‘b’ is the common point to all copies of \( G' \) we fuse Type C label repeatedly for \((x-1)\) times to obtain a labeled copy of \( G'^{(2^x)} \). In structure 3 ‘c’ is the common point to all copies of \( G' \) we fuse Type B label repeatedly for \((x-1)\) times to obtain a labeled copy of \( G'^{(2^x)} \). To obtain labeled copy of \( G' \) we fuse Type D label at respective point on it (vertex a for structure 1 or vertex b for structure 2 or vertex c for structure 3) with \( G' \). We use type B label when \( k \) is of type \( 2x \). For all three structures we use Type A label on \( G \) when \( k \) is of type \( 2x+1 \) for \( x = 0,1,2 \). We use type B label when \( k \) is of type \( 2x \), \( x = 1,2 \). In all the three structures the label number distribution is \( v(0,1) = (3+6x,4+6x) \); \( e(0,1) = (4+7x,3+7x) \) for \( k = 2x \). And when \( k = 2x+1 \) we have \( v(0,1) = (6x,6x+1) \); \( e(0,1) = (7x,7x) \).

This shows that the graph is product cordial. Also the function \( f \) defined is not vertex sensitive in the sense that the same function \( f \) works for all structures.

**Theorem 4.5** Let \( G' \) be a graph obtained from cycle \( C_3 \) by attaching two pendant vertices each at two vertices of \( C_3 \). Let \( G \) be the one point union of \( G' \) is given by \( G = G'^{(k)} \). Then \( G \) is product cordial. 

**Proof:** There are three possible pair wise non-isomorphic structures on \( G' \). Structure 1 is obtained if vertex a is the common point on \( G \). Structure 2 is obtained if vertex b is the common point on \( G \). Structure 3 is obtained if vertex a is the common point on \( G \). This is shown in figure 4.18. Define \( f: V(G) \rightarrow \{0,1\} \) as follows. On using \( f \) we get labeled copies of \( G' \) as Type A and Type B. Both are product cordial. For all three structures we use Type A label on \( G \) when \( k \) is of type \( 2x+1 \) for \( x = 0,1,2 \). We use type B label when \( k \) is of type \( 2x \), \( x = 1,2 \). In all the three structures the label number distribution is \( v(0,1) = (3+6x,4+6x) \); \( e(0,1) = (4+7x,3+7x) \) for \( k = 2x \). And when \( k = 2x+1 \) we have \( v(0,1) = (6x,6x+1) \); \( e(0,1) = (7x,7x) \).
Conclusions: In this paper one point union of $C_3$ related graphs are discussed and are shown to be product cordial. We show that

1) $G' = FL(C_3)$. Then $G = G'(k)$ is product cordial iff $k$ is congruent to $(2 \mod 4)$.  
2) $G' = bull(C_3)$. Then $G = G'(k)$ is product cordial iff $k$ is congruent to $(2 \mod 4)$.  
3) One point union of $k$ copies of $C_3^*$, $G = (C_3^*)^k$ is product cordial graph for all $k$ provided the point common to all copies is the pendent vertex and if common point is degree three then $k$ is equal to 1 or even number only.

4) Let $G'$ be $Tail(C_3, 2P_2)$ obtained from attaching two pendent edges at a vertex of $C_3$, then $G = (G')^k$ is product cordial for all $k$ and all pairwise non-isomorphic structures obtained by taking different vertices on $G'$ as common point.

5) $G'$ be a graph obtained from cycle $C_3$ by fusing two pendent vertices each at each vertex of $C_3$. $G = G'^{(k)}$ is product cordial on all structures.

Thus it is interesting to study the cycles with pendent edges fused at some or all vertices of it for product cordiality.

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