PRIME – ANTIMAGIC LABELING OF GRAPHS

K.Thirugnanasambandam¹, G.Chitra²

¹Assistant Professor, ²Assistant Professor
¹P.G and Research Department of Mathematics
¹Muthurangam Govt. Arts College, Vellore, 632002, India.
²Department of Mathematics, D.K.M.College for Women, Vellore – 632001, India.

Abstract: A Prime graph which admits antimagic labeling is called a Prime - Antimagic labeling. A Prime graph which admits odd antimagic labeling is called a Prime- odd Antimagic labeling. In the present work we investigate Prime - Antimagic labeling of Paths, Caterpillar, Spider Odd Cycles, Complete bipartite graphs, Crown and Prime- odd Antimagic labeling of Paths, Cycles, Complete bipartite graphs, Comb. Also we investigate Strongly Prime - Antimagic labeling of some special graphs of Ladder, Triangular Snake, Quadrilateral Snake, Helm and Gear graphs.

Keywords: Prime - Antimagic labeling, prime-odd antimagic labeling, Strongly prime – antimagic labeling, Path, Caterpillar, Spider, Odd Cycles, Complete bipartite graphs, Combs, Ladder, Triangular Snake, Quadrilateral Snake, Helm, Gear graphs.

AMS Subject Classification (2010):05C78.

1. Introduction

We begin with simple, finite, undirected and non-trivial graph G = (V, E) with the vertex set V and the edge set E. The number of elements of V, denoted as |V| is called the order of the graph G while the number of elements of E, denoted as |E| is called the size of the graph G. In the present work Cn denotes the cycle with n vertices and Pn denotes the path of n vertices. We will give brief summary of definitions which are useful for the present investigation.

Definition 1.1
If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling. S.K. Vaidya and U.M.Prajapati [8] and [9] introduced the prime labeling of a graph which is defined as follows.

Definition 1.2
A Prime labeling of a graph G is an injective function f:V(G)→{1,2,…|V|} such that every pair of adjacent vertices u and v, g.c.d.(f(u),f(v)) = 1. The graph which admits prime labeling is called Prime graph.

Definition 1.3
N.Hartsfield and G.Ringel [1] introduced the concept of antimagic labeling and is defined as follows.

Definition 1.4
A walk in a graph is called a Path in which both the vertices and edges are distinct.

Definition 1.5
Caterpillar is a tree with all vertices either on a single central path or distance one away from it. The central path may be considered to be the largest path in the caterpillar, so that both end vertices have valency one.

Definition 1.6
A Spider SP(Pn,2) is a caterpillar S(X₁,X₂,…..Xₙ) where Xₙ = 2 and X_i = 0 , i = 1,2…n-1.

Definition 1.7
The graph Pn ⊕ K₁ is called a Comb.
Definition 1.8
The graph $C_n \boxtimes mK_i$ is a unicyclic graph with $p = q = n$ $(m+1)$ obtained from the cycle $C_n$ by attaching $m$ – pendant edges at each vertex of the cycle $C_n$.

Definition 1.9
If $m = 1$ the graph $C_n \boxtimes m K_i$ is called a crown.

Definition 1.10 (Bertrand’s Postulate)
For every positive integer $n > 1$ there is a prime $p$ such that $n < p < 2n$.

The present work is aimed to discuss some new families of Prime–Antimagic graphs.

2. Results on Prime-Antimagic labeling
In this section we introduce the concept of Prime–Antimagic labeling and we discuss the Prime–Antimagic labeling of some special graphs.

Definition 2.1
A Prime-Antimagic labeling of a graph $G$ is an bijective function $f: V(G) \rightarrow \{1, 2, \ldots, |V|\}$ such that every pair of adjacent vertices $u$ and $v$, $\gcd(f(u), f(v)) = 1$ and the induced mapping $f^*: E(G) \rightarrow \mathbb{N}$ defined by $f^*(e = uv) = \sum f(u, v)$ where $(u, v) \in E(G)$ is injective and all these edge labelings are distinct.

Theorem 2.2
Every Path $P_n$, $n \geq 2$, admits on Prime-Antimagic labeling.

Proof:
Define $f$ on $V(P_n)$ by $f(v_i) = i$, $i = 1, 2, \ldots, n$ such that every pair of adjacent vertices $u$ and $v$, $\gcd(f(u), f(v)) = 1$. The induced function $f^*: E(G) \rightarrow \mathbb{N}$ defined by $f^*(e = uv) = 2i+1$ for $i = 1, 2, \ldots, n-1$. All these edge labelings are distinct. Thus $P_n$ has Prime-Antimagic labeling.

Theorem 2.3
The Caterpillar $S(X_1, X_2, \ldots, X_n)$ has Prime-Antimagic labeling.

Proof:
The path vertices are denoted as $v_1, v_2, \ldots, v_n$ and the end vertices are denoted as $u_1, u_2, \ldots, u_n$. The assignment of vertex labels are $f(v_i) = 2i-1$, $i = 1, 2, \ldots, n$. and $f(u_i) = 2i$, $i = 1, 2, \ldots, n$ and the condition of prime labeling is every pair of adjacent vertices $u$ and $v$, $\gcd(f(u), f(v)) = 1$. The induced edge labels are $f^*: E(G) \rightarrow \mathbb{N}$ and all are distinct. This completes the proof.

Theorem 2.4
Every Spider $SP(n, 2)$ admits Prime-Antimagic labeling.

Proof:
Define $f: V(G) \rightarrow \{1, 2, \ldots, n+2\}$ by $f(v_i) = n+3-i$, $i = 1, 2, \ldots, n-2$. 
\[ f(v_n) = 3 \]
\[ f(v_{n-1}) = 1 \] and $f(u_1) = 2$, $f(u_2) = 4$ and every pair of adjacent vertices $u$ and $v$, $\gcd(f(u), f(v)) = 1$. The resulting edge labels are $\{3, 4, 5, 8, 11, 13, 15, \ldots, 2q+1\}$ and all are distinct. Thus proved.

Theorem 2.5
The Cycles $C_n$, $n$ – odd admits on Prime-Antimagic labeling.

Proof:
Define $f: V(C_n) \rightarrow \{1, 2, \ldots, n\}$ by $f(v_i) = i$, $i = 1, 2, \ldots, n$ and also every pair of adjacent vertices $u$ and $v$, $\gcd(f(u), f(v)) = 1$. The induced edge labels are defined by $f^*(e = uv) = 2i+1$, $i = 1, 2, \ldots, n-1$ and 
\[ f^*(f(v_i)) = n+1. \]
All the edge labels are distinct and hence the theorem follows.

Theorem 2.6
$K_{1, n}$ is Prime – Antimagic labeling.

Proof:
Define $f: V(G) \rightarrow \{1, 2, \ldots, p\}$ by $f(a_i) = 1$, $f(u_i) = i+1$, $i = 1, 2, \ldots, n$, such that every pair of adjacent vertices $u$ and $v$, $\gcd(f(a_i), f(u_i)) = 1$.

The edge labels are $f: E(G) \rightarrow \{3, 4, \ldots, p+1\}$. All the edge labels are distinct and hence $K_{1, n}$ is Prime – Antimagic labeling.
Theorem 2.7
K \_{m,n} is not Prime – Antimagic labeling if m \geq 2.

Proof:
Define f : V (G) → \{1, 2, ..., p\} such that every pair of adjacent vertices u and v, g.c.d(f(u),f(v)) = 1. This is not possible because it is an incomplete bipartite graph.

Theorem 2.8
The crown C_n⊙K_1 is Prime – Antimagic labeling.

Proof:
Let G = C_n⊙K_1. Here |V (G)| = |E (G)| = 2n.
Define f : V (G) → {1,2,....2n } by f(v_i) = 2i – 1, i = 1,2,……n and f(u_i) = 2i, i = 1,2,….n
and also g.c.d (f(v_i), f (v_{i+1})) = 1, g.c.d (f(v_i), f (u_i)) = 1.
Then the resulting distinct induced edge labelings are defined as follows.
A = f^*(v_{i+1}) = 4i, i = 1, 2,….n-1.
B = f^*(v_n v_1) = 2n
C = f^*(v_i u_1) = 4i – 1, i = 1, 2, n. These sets are disjoint.
Hence C_n⊙K_1 is Prime – Antimagic.

Observation 2.9
The Complete graph K_{1,n} is not Prime – Antimagic if n \geq 4.

Observation 2.10 The friendship graph F_n is not Prime – Antimagic.

3 .Prime – odd Antimagic labeling

In this section we introduce the another new concept of Prime – odd Antimagic labeling and Prime – odd Antimagic labeling of the path P_n, n \geq 2, the cycle C_n, n \geq 3, the star K_{1,n} and the spider SP(P_{n,2}).

Definition 3.1
A connected graph G with |V| = p vertices and |E| = q edges is said to be Prime – odd Antimagic labeling if there is a bijection f:V(G)→{1,3,...,2|V|-1} such that every pair of adjacent vertices u and v, g.c.d(f(u),f(v)) = 1 and the induced mapping f^*:E(G)→N defined by f^*(e = uv) =\sum f(u,v) where (u,v) ∈ E(G) is injective and all these edge labelings are distinct.

Theorem 3.2
Every Path P_n, n\geq 2, admits on Prime- odd Antimagic labeling.

Proof:
Define f on V(P_n) by f(v_i) = 2i-1, i = 1,2,....n, such that every pair of adjacent vertices v_i and v_{i+1}, g.c.d (f(v_i),f(v_{i+1})) = 1.
The induced function f^*: E (G) → N defined by f^*(v_{i+1}) = 4i, for i=1,2,....n-1.
All these edge labelings are distinct and from this the theorem follows.

Theorem 3.3
Every cycle C_n, n\geq 3 admits on Prime – odd Antimagic labeling.

Proof:
Define f:V(G)→{1,3,...,2|V|-1}by f(v_i) = 2i-1, i = 1,2,....n and also satisfies g.c.d (f(v_i), f (v_{i+1})) = 1.
The distinct edge labels are f^*(v_i v_{i+1}) = 4n, i=1,2,...,n-1 and f^*(v_n v_1) =2n.
Thus the Cycle C_n, n \geq 3 is Prime – odd Antimagic labeling.

Theorem 3.4
K_{1,n} is Prime – odd Antimagic labeling.

Proof:
Define f:V(G)→{1,3,...,2|V|-1}by f(a_i) = 1, f(u_i) = 2i+1, i=1,2,....n and g.c.d (f(a_i),f(a_{i+1})) = 1. The distinct edge labels are defined by f^*(a_i u_i) = 2i+2 , i=1,2,...,n and the theorem follows.

Theorem 3.5
Every Spider SP (P_{n,2}) admits on Prime-odd Antimagic labeling.

Proof:
Define f:V(G)→{1,3,...,2|V|-1}by f(u_i) = 2i-1, i=1,2,....n+2.
The induced edge labels are distinct. Hence the condition of antimagic is satisfied.

**Theorem 3.6**
Any Comb $P_n$ $\bigcup K_1$ is Prime – odd Antimagic labeling.

**Proof:**
Define a function $f : V \left( P_n \bigcup K_1 \right) \rightarrow \{1,3,\ldots,2\left|V\right|-1\}$ by

$$f(v_i) = \begin{cases} 4i - 1, & \text{if } i \text{ is odd} \\ 4i - 3, & \text{if } i \text{ is even} \end{cases}$$

and also every pair of adjacent vertices $v_i$ and $v_{i+1}$, g.c.d $\left(f(v_i), f(v_{i+1})\right) = 1$.

Then the induced edge labels of comb as follows.

$$f^*(v_iv_{i+1}) = 8i - 1, \quad 1 \leq i \leq n - 1.$$ 

The resulting edge labels are distinct and the theorem is satisfied.

**4. Strongly Prime Antimagic Graphs**
In this section we introduce the concept of strongly prime antimagic graph labeling. Prime labeling of some classes of graph were discussed by Vaidya.S.K. Prajapati.U.M in [8] and [9]. Prime labeling in the context of some graph operation was discussed in [4] and [5]. Many researchers have studied prime graphs. For e.g. Fu [2] proved that $P_n$ and $K_{1,n}$ are prime graphs. We will give brief summary of definitions which are useful for the present investigations.

**Definition 4.1**
A Graph $G$ is said to be a Strongly Prime graph if for any vertex $v$ of $G$ there exists a Prime labeling $f$ satisfying $f(v) = 1$.

**Definition 4.2**
A connected graph $G$ with $\left|V\right| = p$ vertices and $\left|E\right| = q$ edges is said to be **Strongly Prime Antimagic labeling** if there is an bijection $f: V(G) \rightarrow \{1,3,\ldots,2\left|V\right|-1\}$ such that every pair of adjacent vertices $u$ and $v$ g.c.d $\left(f(u), f(v)\right) = 1$ and if for any vertex $v$ of $G$ such that $f$ satisfying $f(v) = 1$ and the induced mapping $f^*: E(G) \rightarrow \mathbb{N}$ defined by $f^*(e=uv) = \sum f(u,v)$ where $(u,v) \in E(G)$ is injective and all these edge labelings are distinct.

**Definition 4.3**
A Quadrilateral Snake $Q_n$ is obtained from a path $\{u_1 , u_2 , \ldots u_n\}$ by joining $u_i$ and $u_{i+1}$ to two vertices $v_i$ and $w_i$, $1 \leq i \leq n-1$ respectively and then joining $v_i$ and $w_i$.

**Definition 4.4**
The Product $P_n \times P_n$ is called a Ladder and it is denoted by $L_n$.

**Definition 4.5**
The Corona of two graphs $G_1$ and $G_2$ is the graph $G = G_1 \bigcirc G_2$ formed by taking one copy of $G_1$ and $\left|V(G_1)\right|$ copies of $G_2$ where the $i$th vertex of $G_1$ is adjacent to every vertex in the $i$th copy of $G_2$.

**Definition 4.6**
The Helm $H_n$ is a graph obtained from a wheel by attaching a pendant edge at each vertex of the $n$-cycle.

**Definition 4.7**
The gear graph $G_n$ is obtained from the wheel by adding a vertex between each pair of adjacent vertices of the cycle. The gear graph $G_n$ has $2n+1$ vertices and $3n$ edges.

**Definition 4.8**
Triangular Snake $T_n$ is obtained from a path $u_1 , u_2 , \ldots u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $v_i$ for $1 \leq i \leq n-1$, that is every edge of a path is replaced by a triangle $C_3$.

**Theorem 4.9**
The graph $G \bigcirc K_1$ is a Strongly Prime – Antimagic graph where $G = T_n$ for all integer $n \geq 2$.

**Proof:**
Let $\{u_1, u_2, \ldots , u_n\}$ be a path of length $n$. Let $v_i, 1 \leq i \leq n-1$ be the new vertex joined to $u_i$ and $u_{i+1}$. The resulting graph is called $T_n$ and let $x_i$ be the vertex which is joined to $u_i$, $1 \leq i \leq n$, let $y_i$ be the vertex which is joined to $v_i$, $1 \leq i \leq n-1$. The resulting graph is $G_1$ (i.e.) $G \bigcirc K_1$, where $G = T_n$ graph. Now the vertex set of $V(G_1) = \{u_1, u_2, \ldots , u_n, v_1, v_2, \ldots , v_{n-1}, x_1, x_2, \ldots , x_n, y_1, y_2, \ldots , y_{n-1}\}$ and the edge set $E(G_1) = \{u_iu_{i+1}, u_iv_i / 1 \leq i \leq n-1\} \cup \{u_iy_i / 1 \leq i \leq n\} \cup \{v_ix_i, v_iy_i / 1 \leq i \leq n-1\}$. Here $\left|V(G_1)\right| = 4n-2$. Let $v$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case i:**
If $v = u_1$ for some $j \in \{1,2,\ldots , n\}$ then the function $f: V(G) \rightarrow \{1,2,\ldots , 4n-2\}$ defined by
is a Prime labeling for $G_1$ with $f(v) = f(u_i) = 1$.

Case ii:
If $v = x_j$, for some $j \in \{1, 2, \ldots, n\}$ then define a labeling $f_2$ using the labeling $f$ defined in case (i) as $f_2(u_i) = f(x_j) = f(u_i)$ for $j \in \{1, 2, \ldots, n\}$ and $f_2(v) = f(v)$ for all the remaining vertices.

Case iii:
If $v = v_i$, for some $j \in \{1, 2, \ldots, n - 1\}$ then define a labeling $f_1$ using the labeling $f_2$ defined in case (ii) as $f_1(x_j) = f_2(v_i)$ and $f_1(v_i) = f(v)$ for $j \in \{1, 2, \ldots, n - 1\}$ and $f_1(v) = f_2(v)$ for all the remaining vertices.

Case iv:
If $v = y_i$, for some $j \in \{1, 2, \ldots, n - 1\}$ then define a labeling $f_3$ using the labeling $f_1$ defined in case (ii) as $f_3(x_j) = f_1(v_i)$ and $f_3(v_i) = f(v_i)$ for $j \in \{1, 2, \ldots, n\}$ and $f_3(v) = f_1(v)$ for all the remaining vertices.

Thus from all the cases described above $G_1$ is a Strongly Prime graph.

Now to find the edge labels $f^*(e = uv) = \sum f(u, v)$ where $(u, v) \in E(G)$ is injective and all these edge labels are distinct.

Hence $G_1$ admits Prime – Antimagic labeling. Therefore $G_1$ is an Strongly Prime antimagic graph.

**Theorem 4.10**
The graph $G\bar{K}_4$ is a Strongly Prime – Antimagic graph where $G = Q_n$ for all integer $n \geq 2$.

**Proof:**
Let $[u_1, u_2, \ldots, u_n]$ be a path of length $n$. Let $v_i, w_i$ be two vertices joined to $u_i$ and $u_{i+1}$ respectively and then join $v_i$ and $w_i$, $1 \leq i \leq n-1$. The resulting graph is called a quadrilateral snake $Q_n$. Let $x_i$ be the vertex which is joined to $u_i$, $1 \leq i \leq n$, let $y_i$ be the new vertex which is joined to $v_i$, $1 \leq i \leq n-1$ land let $z_i$ be the new vertex which is joined to $w_i$, $1 \leq i \leq n-1$. The resulting graph is $G_1$ (i.e) $G\bar{K}_4$ where $G = Q_n$ graph. Now the vertex set of $V(G_1) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n-1}, w_1, w_2, \ldots, w_{n-1}, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n-1}, z_1, z_2, \ldots, z_{n-1}\}$ and the edge set $E(G_1) = \{(u_i, v_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(w_i, u_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(u_i, v_i) \mid 1 \leq i \leq n\} \cup \{(v_i, w_i) \mid 1 \leq i \leq n\}$. Here $|V(G_1)| = 6n - 4$. Let $v$ be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

**Case i:**
If $v = u_i$ for some $j \in \{1, 2, \ldots, n\}$ then the function $f : V(G_1) \to \{1, 2, \ldots, 6n - 4\}$ defined by

$$f(u_i) = \begin{cases} 6i - 6j + 1 & \text{if } i = j, j + 1, \ldots, n \end{cases}$$

$$f(v_i) = \begin{cases} 6n + 6i - 6j - 3 & \text{if } i = 1, 2, \ldots, j - 1 \end{cases}$$

$$f(w_i) = \begin{cases} 6i - 6j + 3 & \text{if } i = j, j + 1, \ldots, n - 1 \end{cases}$$

$$f(x_i) = \begin{cases} 6n + 6i - 6j + 1 & \text{if } i = 1, 2, \ldots, j - 1 \end{cases}$$

$$f(z_i) = \begin{cases} 6i - 6j + 5 & \text{if } i = j, j + 1, \ldots, n - 1 \end{cases}$$

is a Prime labeling for $G_1$ with $f(v) = f(u_i) = 1$.

**Case ii:**
If $v = x_j$, for some $j \in \{1, 2, \ldots, n\}$ then define a labeling $f_2$ using the labeling $f$ defined in case (i) as $f_2(u_i) = f(x_j) = f(u_i)$ for $j \in \{1, 2, \ldots, n\}$ and $f_2(v) = f(v)$ for all the remaining vertices.
Case iii:
If \( v = v_j \) for some \( j \in \{1,2,\ldots,n-1\} \) then define a labeling \( f_3 \) using the labeling \( f_2 \) defined in case (ii) as \( f_2(x_j) = f_2(v_j) + 4j \) for \( j \in \{1,2,\ldots,n-1\} \) and \( f_2(v) = f_2(v) + 4j \) for all the remaining vertices.

Case iv:
If \( v = w_j \) for some \( j \in \{1,2,\ldots,n-1\} \) then define a labeling \( f_4 \) using the labeling \( f_3 \) defined in case (iii) as \( f_3(w_j) = f_3(w_j) + f_3(v) \) for \( j \in \{1,2,\ldots,n-1\} \) and \( f_3(v) = f_3(v) \) for all the remaining vertices.

Case v:
If \( v = z_j \) for some \( j \in \{1,2,\ldots,n-1\} \) then define a labeling \( f_5 \) using the labeling \( f_4 \) defined in case (iv) as follows:
\[
f_5(z_j) = f_5(w_j), f_5(w_j) = f_5(z_j) \quad \text{for} \quad j \in \{1,2,\ldots,n-1\} \quad \text{and} \quad f_5(v) = f_5(v) \quad \text{for all the remaining vertices.}
\]

Case vi:
If \( v = y_j \) for some \( j \in \{1,2,\ldots,n-1\} \) then define a labeling \( f_6 \) using the labeling \( f_5 \) defined in case (ii) as \( f_5(u_j) = f_5(v_j) \), \( f_5(x_j) = f_5(y_j) \) and \( f_5(x_j) = f_5(x_j) \) for \( j \in \{1,2,\ldots,n-1\} \) and \( f_5(v) = f_5(v) \) for all the remaining vertices.

Thus from all the cases described above \( G_1 \) is a Strongly Prime graph.

Now to find the edge labels \( f'(e = uv) = \sum f(u,v) \) where \( u,v \in E(G) \) is injective and all these edge labelings are distinct.

Hence \( G_1 \) is an Strongly Prime – Antimagic graph.

Theorem 4.11

The graph \( G \Theta_k \) is strongly prime antimagic graph with \( G = L_n \) for all integer \( n \geq 2 \).

Proof:
Let \( G \) be the ladder with vertices \( \{u_1, u_2, \ldots, v_1, v_2, \ldots, v_n \} \). Let \( u'_1 \) be the new vertex joined to \( u_1 \), \( 1 \leq i \leq n \) and \( v'_i \) be the new vertex joined to \( v_i \), \( 1 \leq i \leq n \) in \( G \). The resulting graph \( G_1 = G \Theta_k \) where \( G = L_n \) graph. Now the vertex set \( V(G_1) = \{u_1, u_2, \ldots, v_1, v_2, \ldots, u'_1, v'_1, v'_2, \ldots, v'_n \} \).
The edge set \( E(G_1) = \{u_i u_i', u_i u_i', v_j v'_j, v_j v'_j, 1 \leq i \leq n \} \cup \{u_i v_j, u_i v_j, v_j v_i, v_j v_i' \leq i \leq n \} \). Here
\[
|V(G_1)| = 4n. Let \( v \) be the vertex for which we assign label 1 in our labeling method. Then we have the following cases.

Case (i):
If \( v = u_i \) for some \( i \in \{1,2,\ldots,n\} \) then the function \( f : V(G_i) \rightarrow \{1,2,\ldots,4n\} \) defined by \( f(u_i) = \begin{cases} 
4n + 4i - 4j + 1 & \text{if } i = 1,2,\ldots,n-1 \n4i - 4j + 1 & \text{if } i = 1,2,\ldots,n-1 \n4i + 4j - 4k + 2 & \text{if } i = 1,2,\ldots,n-1 \n4n + 4i - 4k + 4 & \text{if } i = 1,2,\ldots,n-1 \n4n - 1 & \text{if } 4m - 1 \text{ is multiple of } 3 \n2n - 1 & \text{if } 4m - 1 \text{ is multiple of } 3 \n4n & \text{otherwise.} \end{cases} \]

Case (ii):
If \( v = u'_i \) for some \( i \in \{1,2,\ldots,n\} \) then define labeling \( f_2 \) using the labeling \( f \) defined in case (i) as follows:
\[
f_2(u_i) = f(u_i), \quad f_2(u'_i) = f(u_i) \quad \text{for} \quad i \in \{1,2,\ldots,n\} \quad \text{and} \quad f_2(v) = f(v) \quad \text{for all the remaining vertices.} \]

Thus the labeling \( f_2 \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( v = u'_i \) in \( G_1 \).

Case (iii):
If \( v = v_i \) for some \( i \in \{1,2,\ldots,n\} \) then define a labeling \( f_3 \) using the labeling \( f_2 \) defined in case (i) as follows:
\[
f_3(u_i) = f(u_i), \quad f_3(u'_i) = f(u'_i), \quad f_3(v_i) = f(v_i) \quad \text{for} \quad i \in \{1,2,\ldots,n\} \quad \text{and} \quad f_3(v) = f(v) \quad \text{for all the remaining vertices.} \]

Thus the labeling \( f_3 \) is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of \( v = v_i \) in \( G_1 \).

Case (iv):
If \( v = v'_i \) for some \( i \in \{1,2,\ldots,n\} \) then define a labeling \( f_4 \) using the labeling \( f_3 \) defined in case (iii) as follows:
\[
f_4(v_i) = f_3(v_i), \quad f_4(v'_i) = f_3(v_i) \quad \text{for} \quad i \in \{1,2,\ldots,n\} \quad \text{and} \quad f_4(v) = f(v) \quad \text{for all the remaining vertices.} \]

Thus in all three cases \( G_1 \) is a strongly prime graph.

And also the vertex labeling \( f(u) = f(v) = \sum f(e) \) and all these labeling are distinct.

Hence \( G_1 \) is strongly prime antimagic graph.

Theorem 4.12

The Helm \( H_n \) is strongly prime antimagic graph.

Proof:
Let \( v_0 \) be the apex vertex \( v_1, v_2, \ldots, v_n \) be the consecutive rim vertices of \( H_n \) and \( v_1, v_2, \ldots, v_n \) be the pendant vertices of \( H_n \). Let \( v \) be the vertex for which we assign label 1 in our labeling method. Then we have the following cases.

Case (i):
If \( v = v_0 \) then the function \( f : V(H_n) \rightarrow \{1,2,\ldots,2n+1\} \) defined as \( f(v_0) = 1, f(v_1) = 2, f(v_2) = 3, f(v_i) = 2i+1 \) if \( 2 \leq i \leq n \) and \( f(v_i) = 2i \) if \( 2 \leq i \leq n \). Then \( f \) is an injection function. For an arbitrary edge \( e = ab \) of \( H_n \) we claim that g.c.d. (\( f(a), f(b) \)) = 1.

JETIR1806794 | Journal of Emerging Technologies and Innovative Research (JETIR) www.jetir.org | 743
Subcase (i): If $e = v_0v_1$ for some $i \in \{2, 3, \ldots, n\}$ then $\gcd(f(v_0), f(v_1)) = \gcd(1, f(v_i)) = 1$.

Subcase (ii): If $e = v_iv_{i+1}$ for some $i \in \{1, 2, \ldots, n-1\}$ then $\gcd(f(v_i), f(v_{i+1})) = \gcd(2i + 1, 2i + 3) = 1$ as $2i+1$; $2i+3$ are consecutive odd positive integers. If $e = v_1v_2$ then $\gcd(f(v_1), f(v_2)) = \gcd(2,5) = 1$ and if $e = v_nv_1$ then $\gcd(f(v_n), f(v_1)) = \gcd(2n + 1, 2) = 1$ as $2n + 1$ is an odd integer.

Subcase (iii): If $e = v_iv_i'$ for some $i \in \{2, 3, \ldots, n\}$ then $\gcd(f(v_i), f(v_i')) = \gcd(2i+1, 2i) = 1$ as $2i+1$, $2i$ are consecutive positive integers and if $e = v_iv_1$ then $\gcd(f(v_i), f(v_1)) = \gcd(2,3) = 1$ as 2 and 3 are consecutive positive integers.

Case (ii): If $v = v_j$ for some $j \in \{1,2,\ldots,n\}$, $v$ is one of the rim vertices then we may assume that $v = v_1$ then define a labeling $f_1$ using the labeling $f$ defined in case(i) as follows: $f_1(v_0) = f(v_1)$, $f_1(v_2) = f(v_0)$ and $f_1(v) = f(v)$ for all other remaining vertices. Clearly $f$ is an injection. For an arbitrary edge $e = ab$ of $G$ we claim that $\gcd(f(a), f(b)) = 1$.

Case (iii): If $v$ is one of the pendent vertices then we may assume that $v = v_i$ for $i = \frac{p-2}{2}$ or $\frac{p-3}{2}$, where $p$ is the largest prime less than or equal to $2n+1$. According to Bertrand’s postulate such a prime $p$ exist with $\frac{2n+1}{2} < p < 2n+1$.

Case (iv): When $n = 3k + 1$ then define a labeling $f_2$ using labeling $f$ case (iii) as follows: $f_2(v_1) = f(v_2)$, $f_2(v_1') = f(v_1)$ and $f_2(v) = f(v)$ for all other remaining vertices. Then $f$ is an injection function. For an arbitrary edge $e = ab$ of $H_n$ we claim that $\gcd(f(a), f(b)) = 1$. Thus in all the possibilities described above $f$ is prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $H_n$. Hence $H_n$ is strongly prime graph.

Hence all these edge labelings are defined by the condition $\gcd(f(u), f(v)) = 1$. Now to find the edge labeling $f^*(e = uv) = \sum f(uv)$. All these edge labelings are distinct. Hence $G_n$ admits a strongly prime - antimagic labeling.

**Theorem 4.13**
The Gear graph $G_n$ is a strongly prime - antimagic graph.

**Proof:**
Let $v_0$ be the apex vertex $v_1, v_2, \ldots, v_n, v_i, v_{i+1}, \ldots v_1$ be the consecutive rim vertices. Let $v$ be an arbitrary vertex of $G_n$ that is $v = v_0$. Then the function $f : V(G_n) \rightarrow \{1, 2, \ldots, 2n+1\}$ defined as $f(v_i) = 2i + 1$ for $i = 0, 1, 2, \ldots, n$, $f(v_1) = 2i + 2$ for $i = 1, 2, \ldots, n-1$, $f(v_0) = 2$. Clearly $f$ is an injection. For an arbitrary edge $e = ab$ of $G_n$ we claim that $\gcd(f(a), f(b)) = 1$.

Now to find the edge labeling $f^*(e = uv) = \sum f(uv)$. All these edge labelings are distinct. Hence $G_n$ admits a strongly prime - antimagic labeling.

5. **Conclusion**
In this article we have investigated Prime-Antimagic labeling of Paths, Caterpillar, Spider, Odd Cycles, Complete bipartite graphs, Crown and Prime- odd Antimagic labeling of Paths. Cycles, Complete bipartite graphs, Spider and Comb. Also we investigated strongly prime – antimagic labeling of Triangular snake, Quadrilateral snake, Ladder, Helm, Gear graphs. Analogous work can be carried out for other families and in the context of different types of graph labeling techniques.

**References**