Vertex Prime Invariance Of some Path Union Graphs

Mukund V. Bapat

1. Abstract: we investigate path unions of $C_3$, $W_4$, and $W_4^*$ for vertex prime labeling. All different non isomorphic structures of these path unions are shown to be vertex prime. This property of graphs is called as invariance under labeling.

2. Key words: labeling, vertex prime, wheel, path union, crown.

2. Introduction:

The graphs we consider are finite, connected, simple and un-directed. We refer F. Harary[], Dynamic survey of graph labeling [3] for definitions and terminology. Deretsky, Lee, Mitchem Proposed a labeling called as vertex prime labeling of graph[2]. A function $f : E(G) \rightarrow \{1,2,\ldots,|E|\}$ is such that for any vertex $v$ the gcd of all labels on edges incident with $v$ is 1. This is true to all vertices with degree at least 2. The graph that admits vertex prime labeling is called as vertex prime graph. They have shown that all forests, connected graphs, $5C_{2m}$, graph with exactly two components one of which is not odd cycle etc are vertex prime. One should refer A Dynamic survey of graph labeling by Joe Gallian [3] to find further work done in this type of labeling.

In this paper we discuss vertex prime labeling of path union graphs. Path union $P_m(G)$ is obtained by taking $m$ copies of graph $G$, identify a fixed same vertex from one copy of $G$ with vertex of $P_m$. If we change the vertex on $G$ used to identify with vertex of $P_m$ we may will get a different structure of path union.

4. Preliminaries

4.1 Crown of $(p,q)$ graph $G$ denoted by $G^+$ is obtained by attaching a pendent edge to each vertex of $G$. It has $q + p$ edges and $2p$ vertices. A double crown $G^{++}$ is obtained by attaching 2 pendent edges at each vertex of $G$. etc.

4.2 A wheel graph $W_n$ is obtained by taking a cycle $C_n$ and a new vertex $w$ outside of $C_n$. $W_n$ is joined to each vertex of $C_n$ by an edge each. It has $2n$ edges and $n+1$ vertices.

4.3 To obtain a path union of $(p, q)$ graph $G$ we fuse a copy of $G$ at a given fixed vertex of $G$ on every vertex of path $P_m$. If we change the vertex on graph $G$ used to fix with vertex of $P_m$ we may will get a different structure of path union.

5. Theorems proved:

5.1 Path union of $C_3$ crown (i.e. $G = P_m(c_3^*)$) is Vertex Prime.

Proof: There are two non-isomorphic structures possible. Structure I in which a degree 3 vertex on $C_3$ crown is used for to identify with vertex of $P_m$. We define $G$ in terms of vertex set and edge set.
Define a function $f: E(G) \rightarrow \{1, 2, \ldots, |E|\}$ as follows:

$$f(e_i) = i, \quad i = 1, 2, \ldots, m - 1, f(c_{ij}) = m - 1 + (i - 1)6 + j, \quad j = 1, 2, 3$$

$$f(b_{ij}) = m + 6i - 4 + j, \quad j = 1, 2, 3, \quad i = 1, 2, \ldots, m.$$
Define a function \( f : E(G) \rightarrow \{1, 2, \ldots, |E|\} \) as follows:

\[
f(e_i) = i, \quad i = 1, 2, \ldots, m - 1.
\]

\[
f(c_{i,j}) = m - 1 + (i - 1)7 + j, \quad j = 1, 2, 3,
\]

\[
f(b_{i,j}) = m + (i - 1)6 \quad \text{for } j = 1.
\]

\[
f(b_{i,j}) = m + 4 + (i - 1)6 \quad \text{for } j = 2,
\]

\[
f(b_{i,j}) = m + 5 + (i - 1)6 \quad \text{for } j = 3.
\]

5.2 Path union of wheel graph (\( W_n \)) is vertex prime for \( n = 4 \)

**Proof:** We define the path union of wheel graph (\( W_4 \)) i.e \( G = P_m(W_4) \) as follows.

\[
V(G) = \{w_1, v_1, v_2, \ldots, v_m\} \cup \{u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}\}
\]

\[
E(G) = \{e_i=(v_i v_{i+1})/i=1, 2, \ldots, m-1\} \cup \{c_{i,j}=(u_{i,j} u_{i,j+1}), j=1, 2, \ldots, n \text{ and } n+1 \text{ taken (mod } n)\}, \text{ here } n = 4\}\cup \{b_{i,j}=(w_i u_{i,j})/i=1, 2, \ldots, m \text{ and } j = 1, 2, 3, 4\}.
\]

Define a function \( f : E(G) \rightarrow \{1, 2, \ldots, q\} \) as follows:
Define a function $f : E(G) \rightarrow \{1,2,\ldots,|E|\}$ as follows:

- $f(e_i) = i$, $i = 1,2,\ldots,m-1$ and $j = 1,2,3,4$.
- $f(c_{i,j}) = m-1 + (i-1)8 + j$, $i = 1,2,\ldots,m-1$ and $j = 1,2,3,4$.
- $f(b_{i,j}) = f(c_{i,4}) + j$, $i = 1,2,\ldots,m-1$ and $j = 1,2,3,4$.

The graph is vertex prime.

We obtain this structure by identifying a 3-degree vertex $u_{i,1}$ on cycle $C_4$ of $W_4$ with vertex $v_i$ of path $P_m$.

- $f(e_i) = i$, $i = 1,2,\ldots,m-1$.
- $f(c_{i,j}) = (m-1) + 8(i-1) + j$, $i = 1,2,3,4$.
- $f(b_{i,j}) = f(c_{i,4}) + j$, $i = 1,2,3,4$.

The resultant graph is vertex prime.
The both of the structures of $P_m(W_4)$ are vertex prime graphs.

5.3 Crown of $W_4$ (i.e. G = $W_4^*$ (all four non isomorphic structures) is vertex prime graph.

Proof: We define the graph $G$ as: (structure 1)

$$V(G) = \{w_i,v_1,v_2,\ldots,v_m\} U \{ u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}\} U \{ w_{i,j}/i=1,2,3,\ldots,m, j=1,2,3,4\}$$

$$E(G) = \{e=(v_{i+1}/i=1,2,\ldots,m-1) U \{c_{i,j}=(u_{i,j}u_{i,j+1}), j=1,2,\ldots,n \text{ and } n+1 \text{ taken (mod } n), \text{ here } n=4\} U \{b_{i,j}=(w_iu_{i,j})/i=1,2,\ldots,m \text{ and } j=1,2,3,4\} U \{q_{i,j}=(u_{i,j}w_{i,j})/i=1,2,3,4 \text{ and } q_{i,5}=(w_iw_{i,5}) \}$$

Define a function $f:E(G) \rightarrow \{1,2,\ldots,q\}$ as follows:

$$f(e_i)=i, \quad i = 1,2,\ldots,m$$

$$f(c_{i,j})=m-1+(i-1)8+j, \quad j = 1,2,\ldots,5, \quad i = 1,2,\ldots,m,$$

$$f(c_{i,j})=m+8(i-1)+j-1, \quad j = 1,2,\ldots,5, \quad i = 1,2,\ldots,m,$$

$$f(b_{i,j})=f(c_{i,4})+j, \quad i=1,2,\ldots,m-1 \text{ and } j=1,2,3,4.$$  

$$f(q_{i,j}) = f(b_{i,4})+j, i=1,2,\ldots,m \text{ and } j=1,2,\ldots,5;$$

To obtain structure 2 we take a path $P_m$ and $m$ copies of $W_4^*$. At each vertex of $P_m$ attach a copy of $W_4^*$ by the pendent vertex at hub $w$. We define this graph as follows.

$$V(G) = \{v_1,v_2,\ldots,v_m\} U \{u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}, w_i\} U \{w_{i,1}, w_{i,2}, \ldots, w_{i,5}\}$$

$$E(G) = \{e=(v_{i+1}/i=1,2,\ldots,m-1) U \{c_{i,1}=(u_{i,1}u_{i,2}), c_{i,2}=(u_{i,2}u_{i,3}), c_{i,3}=(u_{i,3}u_{i,4}), c_{i,4}=(u_{i,4}u_{i,1})\} U \{b_{i,j}=(w_{i}u_{i,j})/i=1,2,\ldots,m \text{ and } j=1,2,3,4,5\} U \{ q_{i,j}=(u_{i,j}w_{i,j})/i=1,2,\ldots,m, j=1,2,3,4 \text{ and } q_{i,5}=(w_iw_{i,5}) \}$$
Define a function as:

\[ f: E(G) \rightarrow \{1, 2, \ldots, |E|\} \]

\[ f(e_i) = i, \quad i = 1, 2, \ldots, m - 1. \]

\[ f(q_{i,1}) = m + 13(i - 1), \quad i = 1, 2, \ldots, m. \]

\[ f(b_{i,j}) = f(q_{i,1}) + j, \quad j = 2, 3, 4, \quad i = 1, 2, \ldots, m. \]

\[ f(c_{i,j}) = f(b_{i,4}) + j, \quad j = 2, 3, 4, \quad i = 1, 2, \ldots, m. \]

\[ f(q_{i,j}) = f(c_{i,4}) + j, \quad j = 1, 2, 3, 4. \]

The resultant labeling is vertex prime.

We define **structure 3** as:

\[ V(G) = \{v_1, v_2, \ldots, v_m\} \cup \{u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}, w_i\} \cup \{w_{i,1}, w_{i,2}, w_{i,3}, w_{i,4}, w_{i,5}\}. \]

\[ E(G) = \{e_i = (v_i, v_{i+1})/i = 1, 2, \ldots, m - 1\} \cup \{c_{i,1} = (u_{i,1}, u_{i,2}), c_{i,2} = (u_{i,2}, u_{i,3}), c_{i,3} = (u_{i,3}, u_{i,4}), c_{i,4} = (u_{i,4}, u_{i,1})\} \cup \{b_{i,j} = (w_{i,j}, w_{i,j})/i = 1, 2, \ldots, m, \]

\[ j = 1, 2, 3, 4.\} \cup \{q_{i,j} = (u_{i,j}, w_{i,j})/i = 1, 2, \ldots, m, j = 1, 2, 3, 4 \} \cup \{q_{i,5} = (w_{i,5}, w_{i,5})\}.\]

The \( P_m(W_4^*) \) is obtained by taking \( m \) copies of \( P_m(W_4^*) \) and identifying \( q_{i,1} \) (it is the pendant vertex at point \( u_{i,1} \)) with \( v_i, \quad i = 1, 2, \ldots, m.\)

Define a function \( f \) as follows:

\[ f: E(G) \rightarrow \{1, 2, \ldots, |E|\} \]

\[ f(e_i) = i, \quad i = 1, 2, \ldots, m - 1. \]

\[ f(q_{i,1}) = m + 13(i - 1), \quad i = 1, 2, \ldots, m. \]

\[ f(b_{i,j}) = f(q_{i,1}) + j, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, 3, 4 \]

\[ f(c_{i,j}) = f(b_{i,4}) + j, \quad j = 2, 3, 4, \quad i = 1, 2, \ldots, m. \]
f(qi,j)=f(b_{i,4})+j, \quad j=2,3,4,5

The resultant labeling is vertex prime.

**Structure 4** is defined as follows. \( V(G) = \{v_1,v_2,\ldots,v_m\} \cup \{u_{i,1},u_{i,2},u_{i,3},u_{i,4},w_i\} \cup \{w_{i,1},w_{i,2},w_{i,3},w_{i,4},w_{i,5}\} \).
\( E(G) = \{e_i=(v_i,v_{i+1})/i=1,2,\ldots,m-1\} \cup \{c_{i,1}=(u_{i,1}u_{i,2}), c_{i,2}=(u_{i,2}u_{i,3}), c_{i,3}=(u_{i,3}u_{i,4}), c_{i,4}=(u_{i,4}u_{i,1})\} \cup \{b_{i,j}=(w_iu_{i,j})/i=1,2,\ldots,m, \text{ and } j=1,2,3,4.\} \cup \{q_{i,j}=(u_{i,j}w_{i,j})/i=1,2,\ldots,m,j=1,2,3,4 \text{ and } q_{i,5}=(w_iu_{i,5})\}.

The \( P_m(w_4^*) \) is obtained by attaching vertex \( u_{i,1} \) to \( v_i \).
f(e_i)=i , i = 1,2,...,m-1.

f(q_{i,1}) = m+13(i-1), i = 1,2,...,m.

f(c_{i,j})=f(q_{i,1})+j, i = 1,2,...m , j=1,2,3,4

f(b_{i,j})=f(c_{i,4})+j, j=2,3,4.,i=1,2,...,m.

f(q_{i,j})=f(b_{i,4})+j= 2,3,4,5

The resultant labeling is vertex prime.

**Conclusions**: There are different structures of path union P_m(G) possible. We have shown that for G = C_3^+, W_4, C_4^+
All possible non isomorphic structures are vertex prime graphs. It is necessary to investigate this property for other graphs also.

References:


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1 Mukund V. Bapat, Hindale, Devgad, Sindhudurg, Maharashtra India: 416630