AN OVERVIEW OF FUZZY SETS

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Abstract: The world where we live is full of ambiguities. In earlier days, there was no mathematical concept to define vagueness. The laws of logic, the law of identity, the law of non-contradiction and the law of excluded middleware introduced and can be applied in any kind situation. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition — an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since the characteristic functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1.In fuzzy set theory, classical bivalent sets are usually called crisp sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise. This paper presents an brief overview of fuzzy sets which cuts across some definitions, operations, properties, operators and extension principle on fuzzy sets.

Keywords: **Crisp sets, extension principle, fuzzy sets, membership functions, type-2 fuzzy sets**

I. INTRODUCTION:-

The evolution of Mathematics started with counting numbers, which are called Natural numbers. The set of natural numbers formed the basis for theories and calculations. The inclusion of zero and negative numbers introduced us to the world of integers, followed by rational numbers, irrational numbers, real numbers, and complex numbers. The idea of fuzzy logic was first advanced by Dr. Lotfi Zadeh of the University of California at Berkeley in the 1960s. He claimed that many sets in the world that surrounds us are defined by a non-distinct boundary. In Fuzzy logic, there are not just two alternatives but a whole continuum of truth values for logical propositions. A proposition A can have truth value 0.7 and its complement A^c the truth value 0.3. In fuzzy logic 0 and 1 are the extreme cases of truth but also includes the various states of truth in between so that, for example, Milk {not sweet, little sweet, sweet and too sweeter} can be written as {0/not sweet; 0.2/little sweet; 0.4/sweet; 1/too sweeter;} Fuzzy logic designed to solve problems in the same way as our brain solve. We aggregate data and form a number of partial truths which we aggregate further into higher truths. In real world, we use fuzzy knowledge, knowledge that is ambiguous, probabilistic, imprecise, uncertain or inexact in nature. There is fuzzy information involved in human thinking and reasoning like young, tall, good, high, cold.

Our systems are not able to answer many questions. The reason is, most systems are designed based upon classical set theory and two valued logic (true, false) which unable to cope with unreliable and incomplete information and give expert opinions. We want our systems also be able to cope with unreliable and incomplete information and give expert opinions. Fuzzy sets have been able to provide solutions to many real world problems.

The theory of fuzzy sets has advanced in a variety of ways and in many disciplines since its inception in 1965. Applications of this theory can be found, for example, in artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management science, operations research, pattern recognition, and robotics. Mathematical developments have advanced to a very high standard and are still forthcoming today. Researchers across the world developed various concepts bridging fuzzy with most of the area in Mathematics and introduced Fuzzy Real line, fuzzy topology, fuzzy trigonometry, etc

II. CRISP SETS AND FUZZY SETS

A fuzzy set extends the binary membership {0,1} of a conventional set to a spectrum in the interval [0,1]. The characteristic function of a crisp set assigns a value of either 1 or 0 to each individual in the universal set, there by discriminating between members and non members of the crisp set under consideration. This function can be generalized such that the value assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements. Such a function is called a membership function and the set defined by it is a fuzzy set,

A key difference between crisp and fuzzy sets is their membership function. A crisp set has a unique membership function whereas a fuzzy set can have infinite number of membership functions to represent it. For fuzzy sets uniqueness is sacrificed, but flexibility is gained because the membership function can be adjusted to maximum the utility for a particular application.

Figures 1.1(a) and 1.1(b) show the difference between the crisp set and fuzzy set boundaries. In Figure 1.1(a) the classical set is well defined by crisp boundaries, here there is no uncertainty in the prescription or location of the boundaries of the set. The boundary of the crisp set is an unambiguous line. In Figure 1.1(b) the boundary of the fuzzy set is shown which is prescribed by vague or ambiguous properties.

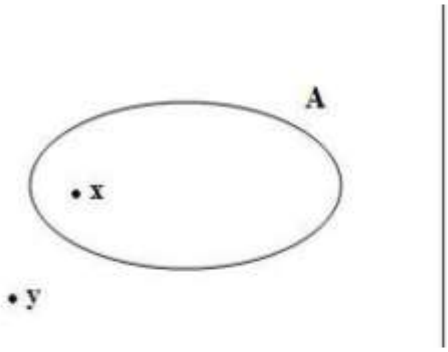


figure 1.1(a) crisp set boundary

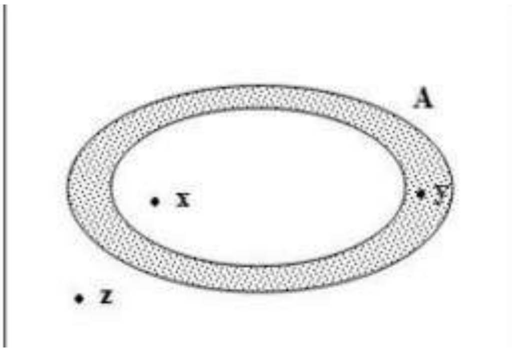


figure 1.1(b) fuzzy set boundary

In Figure 1.1(a) point x is clearly a member of crisp set A and point y is unambiguously not a member of the set A. Figure 1.1(b) shows the vague ambiguous boundary of a fuzzy set A on the universe X. The shaded boundary represents the boundary region of A. Here the point x is clearly a full member of the set, outside the boundary region of the fuzzy set the point z is clearly not a member of the fuzzy set. However the membership of point y which is on the boundary region is ambiguous. If the complete membership in a set (point x) is represented by the number 1 and non-membership in a set (point z) is represented by 0, then the point y must have some intermediate value of membership (partial membership in the fuzzy set) on the interval (0, 1).Presumably the membership of point y in A approaches a value of 1 as it moves closer to central region and approaches a value 0 as it moves closer to leaving the boundary region of A. From Figure 1.1 (b) we have an uncertainty for point y, which is on the boundary region. The membership value of point y is uncertain and the value lies between the interval (0,1).

The advantage of fuzzy set theory is it has the property of relativity, variability and inexactness in the definition of its elements or it entertains imprecise information. Therefore every scientific discipline based on experiments and measurements can make use of fuzzy sets in mathematical modeling and in analytical solutions to improve the generality.

III. MEMBERSHIP FUNCTION(MF)

Membership functions characterize the fuzziness in a fuzzy set —whether the elements in the fuzzy sets are discrete or continuous—in a graphical form for eventual use in the Mathematical formalisms of fuzzy set theory. Because of its mapping characteristics like a function, it is called membership function. The membership function essentially embodies all fuzziness for a particular fuzzy set; its description is the essence of a fuzzy property or operation. Since all information contained in a fuzzy set described by its membership function, it is useful to develop a lexicon of terms to describe various special features of this function.

The membership of an object in a fuzzy set can be approximate. Lotfi Zadeh states that the membership to accommodate various "degrees of membership" on the real continuous interval [0,1] where the end points of 0 and 1 can form to non membership and full membership respectively. But there are infinite number of values in between the end points [0, 1], which can represent various degrees of membership for an element. A crisp set has a unique membership function where as a fuzzy set can have an infinite number of membership functions to represent it. In theory, membership functions usually can take any form. But in most practical applications, triangular, Gaussian and trapezoidal membership functions are commonly used. The membership function is the crucial component of a fuzzy set as the operations with fuzzy sets are defined via their membership function.

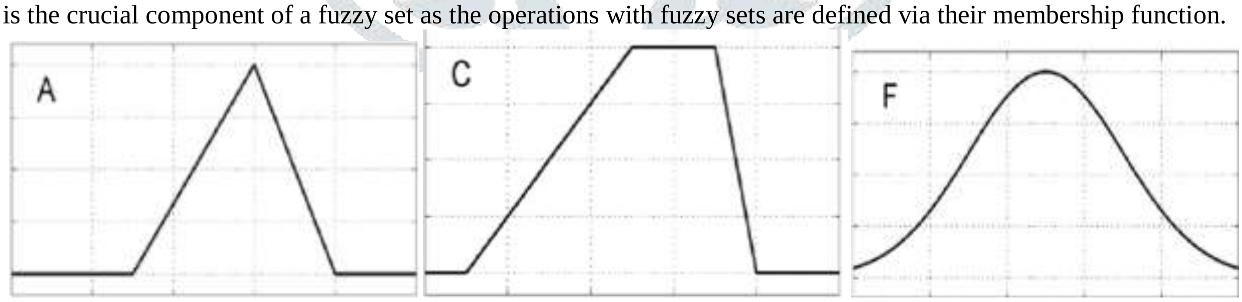


Fig 1.2 (A) triangular C) trapezoidal mf (F) gaussian mf

IV. FUZZY SETS- BASIC DEFINITIONS

Definition 1. (Fuzzy set) Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function

$$\mu_A : X \to [0, 1]$$

and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

It is clear that A is completely determined by the set of tuples

$$A = \{(u, \mu_A(u)) | u \in X\}.$$

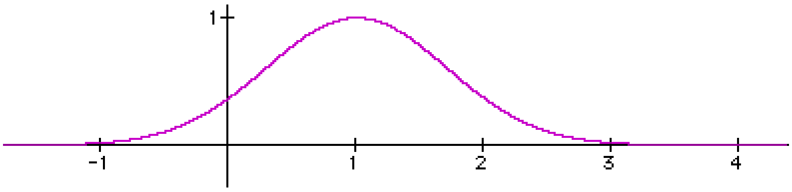
Often we write A(x) instead of $\mu_A(x)$. The family of all fuzzy sets in X is denoted by F(X).

If $X = \{x_1, \ldots, x_n\}$ is a finite set and A is a fuzzy set in X then we often use the notation $A = \mu_1/x_1 +$ ------- $+ \mu_n/x_n$

where the term μ_i/x_i, $i = 1, \ldots, n$ signifies that μ_i is the grade of membership of x_i in A and the plus sign represents the union.

Example 1. The membership function of the fuzzy set of real numbers "close to 1", can be defined as $A(t) = \exp(-\beta(t-1)^2)$

Where β is a positive real number.



A membership function for"x is close to 1".

Definition 2. (support) Let A be a fuzzy subset of X; the support of A, denoted supp(A), is the crisp subset of X whose elements all have non zero membership grades in A.

supp(A) = {x ∈ X|A(x) > 0}.

Definition 3 (normal fuzzy set) A fuzzy subset A of a classical set X is called normal if there exists as x ∈ X such that A(x) = 1. Otherwise A is subnormal.

Definition 4. (α-cut) An α-level set of a fuzzy set A of X is a non-fuzzy set denoted by [A]α and is defined by

$$[A]_\alpha = \begin{cases} \{t \in X \mid A(t) \geq \alpha\} & \text{if } \alpha > 0 \\ cl(suppA) & \text{if } \alpha = 0 \end{cases}$$

where cl(suppA) denotes the closure of the support of A.

Example 3. Assume X = {−2, −1, 0, 1, 2, 3, 4} and A = 0.0/ −2 + 0.3/ − 1 +0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4, in this case

$$[A]_\alpha = \begin{cases} \{-1, 0, 1, 2, 3\} & \text{if } 0 \leq \alpha \leq 0.3 \\ \{0, 1, 2\} & \text{if } 0.3 < \alpha \leq 0.6 \\ \{1\} & \text{if } 0.6 < \alpha \leq 1 \end{cases}$$

Definition 5. (Convex fuzzy set) A fuzzy set A of X is called convex if [A]α is a convex subset of X ∀ α ∈ [0, 1].

Definition 6. (Empty fuzzy set) The empty fuzzy subset of X is defined as the fuzzy subset Φ of X such that Φ (x) = 0 for each x ∈ X. It is easy to see that Φ⊂ A holds for any fuzzy subset A of X.

Definition 7. The largest fuzzy set in X, called universal fuzzy set in X, denoted by 1_X, is defined by $1_X(t) = 1$, ∀ t ∈ X. It is easy to see that A ⊂ 1_X holds for any fuzzy subset A of X.

Definition 8: (Inclusion of fuzzy sets): given two fuzzy sets A and B included in X, the inclusion A ⊆ B takes place iff $\mu_A(x) \leq \mu_B(x)$, ∀ x ∈ X

Definition 9: (Equality of two fuzzy sets): Two fuzzy sets A and B included in X are equal iff $\mu_A(x) = \mu_B(x)$, ∀ x ∈ X. Equivalently, two fuzzy sets A and B included in X are equal iff A ⊆ B and B ⊆ A

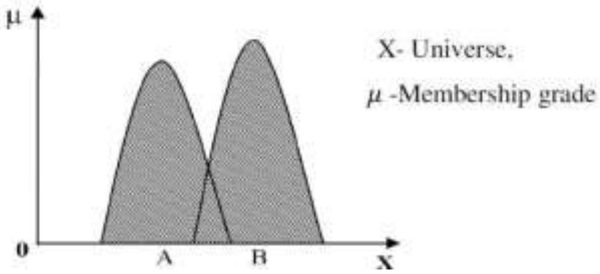
Definition 10: The cardinality or "power" of a fuzzy set A is defined as $|A|=\sum_{x\in X} \mu_A(x)$ and ||A||=|A|/|X|

The relative cardinality of a fuzzy set depends on the cardinality of the universe. So to compare fuzzy sets by their relative cardinality, same universe must be chosen.

V. OPERATIONS ON FUZZY SETS

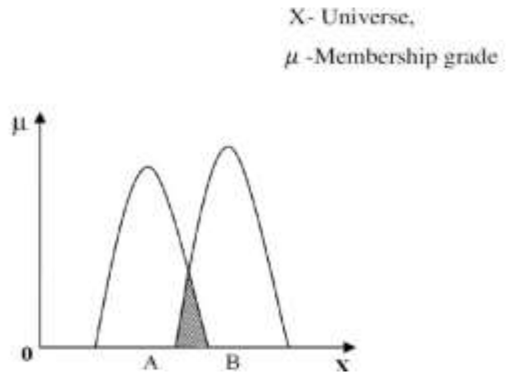
Given two fuzzy sets A = {(x, $\mu_A(x)$)|x ∈ X} and B = {(x, $\mu_B(x)$)|x ∈ X} over the same universe of discourse X

- **the union of the fuzzy sets** A and B as the fuzzy set C = A∪ B, given by C = {(x, $\mu_C(x)$)|x ∈ X},
  where $\mu_C(x) = \max\{(\mu_A(x), \mu_B(x)\}$, x ∈ X



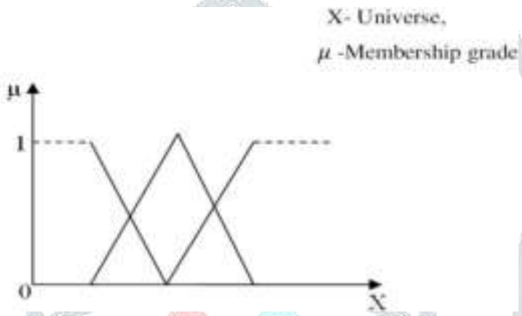
union of two fuzzy sets

- **the intersection of the fuzzy sets** A and B as the fuzzy set D = A ∩B, given by D = {(x, μ_D (x) /x ∈ X},
  where $\mu_D(x) = \min\{(\mu_A(x), \mu_B(x)\}$, x ∈ X



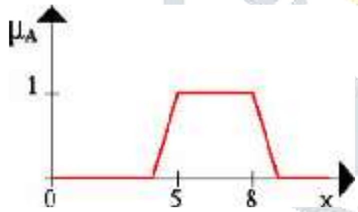
intersection of two fuzzy sets

- **the complement of A in X** as the fuzzy set $E=A^c$, given by $E = \{(x, \mu_E(x)) | x \in X\}$, where $\mu_E(x) = 1 - \mu_A(x), x \in X$

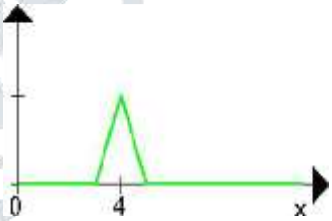


compliment of a fuzzy set

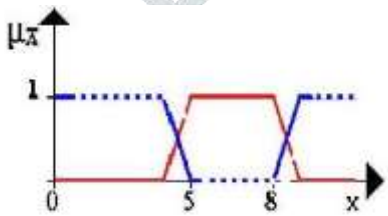
For example: Let A be a fuzzy interval between 5 and 8 and B be a fuzzy number about 4. The corresponding figures are shown below.



fuzzy interval between 5 & 8

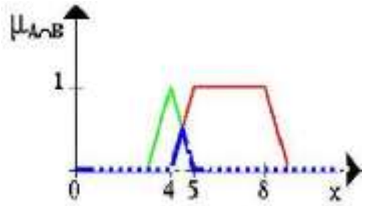


fuzzy number about 4



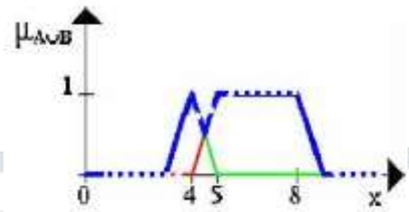
An example for a negation

The dotted line in the figure above is the negation of the fuzzy set A. To negate the fuzzy set A, subtract the membership value in the fuzzy set from one. For example, the membership value at 5 is one. In the negation, the membership value at 5 would be zero (1-1=0). If the membership value is 0.4, in the negation, the membership value would be 0.6 (1- 0.4=0.6).



The fuzzy set between 5 and 8 AND about 4

This time in the above figure minimum criterion is used, the dotted line represents the result. To find the intersection of these sets take the minimum of the two membership values at each point on the x-axis. For example, in the figure A fuzzy set has a membership of zero when x = 4 and B fuzzy set has a membership of one when x = 4. The intersection would have a membership value of ZERO when x = 4 because the minimum of zero and one is zero.



The Fuzzy set between 5 and 8 OR about 4

This time in the above figure the maximum criterion is used. To find the union of these sets take the maximum of the two membership values at each point on the x-axis. For example, A fuzzy set has a membership of ZERO when x = 4 and B fuzzy set has a membership of ONE when x = 4. The union would have a membership value of ONE when x = 4 because the maximum of zero and one is one.

- the membership function of the addition of 2 fuzzy sets A and B is defined by $\mu_{A+B}(z) = \mu_C(z) = \sup_{z=x+y}\{\mu_A(x), \mu_B(y)\}$, x,y ∈ X
- the membership function of the product of 2 fuzzy sets A and B is defined by $\mu_{AB}(z) = \mu_C(z) = \sup_{z=xy}\{\mu_A(x), \mu_B(y)\}$, x,y ∈ X

VI. SOME OPERATORS ON FUZZY EXPRESSION

There are some operators in the fuzzy expression such as ¬ for (negation), ∧ for (conjunction), ∨ for (disjunction), and→for (implication).

Negation "not": $\neg a = 1 - a$

Conjunction "and" "∧" : $a \wedge b = \text{Min}(a, b)$

Disjunction "or" "∨": $a \vee b = \text{Max}(a, b)$

Implication "→": $a \rightarrow b = \text{Min}(1, 1+b - a)$

The following special symbols can also be used to represent fuzzy logic operations and connective structure:

⇒ for " if-then"

⇔ for " if and only if"

VII. PROPERTIES OF FUZZY SETS

The membership values of a crisp set are a subset of the interval [0,l], because of this fact classical sets can be thought of as a special case of fuzzy sets.

Frequently used properties of fuzzy sets are listed below:

- **Commutativity**

  $\mu_{A\cup B} = \mu_{B\cup A}$ and $\mu_{A\cap B} = \mu_{B\cap A}$
- **Associativity**

  $\mu_{A\cup(B\cup C)} = \mu_{(A\cup B)\cup C}$ and $\mu_{A\cap(B\cap C)} = \mu_{(A\cap B)\cap C}$
- **Distributivity**

  $\mu_{A\cup(B\cap C)} = \mu_{(A\cup B)\cap(A\cup C)}$ and $\mu_{A\cap(B\cup C)} = \mu_{(A\cap B)\cup(A\cap C)}$
- **Idempotency**

  $\mu_{A\cup A} = \mu_A$ and $\mu_{A\cap A} = \mu_A$
- **Identity**

  $\mu_{A\cup\Phi} = \mu_A$ and $\mu_{A\cap X} = \mu_A$

  $\mu_{A\cap\Phi} = \mu_\Phi$ and $\mu_{A\cup X} = \mu_X$
- **Involution**

  $\mu_{(A^c)^c} = \mu_A$
- **Demorgan 's law**

$\mu_{(A\cup B)^c} = \mu_{A^c \cap B^c}$ and $\mu_{(A\cap B)^c} = \mu_{A^c \cup B^c}$

VIII. TYPES OF FUZZY SETS

Several types of fuzzy sets have been proposed in literature. When the membership function does not assign to each element of the universal set one real number but a closed interval of real numbers, we get Interval Valued Fuzzy Sets. These sets are defined formally by μ_A: X→∈[0,1] where ∈ [0,1] denotes the family of all closed intervals of real numbers in [0,1]. The primary disadvantage of interval valued fuzzy sets is that the processing of interval valued fuzzy sets is computationally more demanding. When interval valued fuzzy sets are further generalized by allowing their intervals to be fuzzy, we get fuzzy sets of type 2. Their membership functions have the form μ_A: X→ F [0,1], where F [0,1] denotes the set of all ordinary fuzzy sets that can be defined within the universal set [0,1]. Fuzzy sets of still higher types could be obtained recursively in the same way. When fuzzy sets are defined within a universal set whose elements are ordinary fuzzy sets, these fuzzy sets are known as Level 2 Fuzzy Sets. Their membership functions have the form μ_A: F [X] → [0,1],where F [X] denotes the fuzzy power set of *X*. Level 2 fuzzy sets can be generalized into level 3 fuzzy sets and further on.

IX. EXTENSION PRINCIPLE:

Definition: Let X, Y be universes, A ⊂ X be a fuzzy set in X and let f: X → Y be a function such that y = f (x).

The extension principle allows the definition of a fuzzy set B ⊂ Y,

B = {(y, μ_B(y)) | y = f (x), x ∈ X}, where:

$$\mu_B(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x) & \text{if } (\exists)\, f^{-1}(y) \\ 0, & \text{otherwise} \end{cases}$$

Explanation

If we have a fuzzy set A ⊂ X and a function f : X → Y , the extension principle allow us to determine the fuzzy set B ⊂ Y which is the image (the mapping) of the set A through the function f . The following situations may appear:

- If an element y ∈ Y is the image of a unique element x ∈ X, then it is straight forward to consider $\mu_B(y) = \mu_A(x)$
- If an element y ∈ Y is the image of no one element x ∈ X, then it would be normal to consider that $\mu_B(y) = 0$
- If an element y ∈ Y is the image of several elements x ∈ X (for example x_i , x_j, . . . , x_k), then, the degree of membership of y to B will be the maximum between the degrees of membership of the elements x_i,x_j , . . . , x_k to A

In formula, by x ∈ f^{-1}(y) are denoted those elements x form X whose image through function f is y from Y, i.e. y = f (x).

The extension principle in its general form

Definition: Let X = $X_1 \times X_2 \times \ldots \times X_n$ be the cartesian product of the universes X_i, i = 1, . . . , n, and $A_i \subset X_i$ be fuzzy sets in X_i , and let f : X → Y be a function such that y = f (x_1, x_2, . . . , x_n), $x_i \in X_i$, i = 1, . . . , n, and Y is also an universe.

The extension principle permits the definition of a fuzzy set B ⊂ Y ,

B = {(y, μ_B(y)) | y = f (x_1, x_2, . . . , x_n), (x_1, x_2, . . . , x_n) ∈ X},

where:

$$\mu_B(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \min(\mu_{A1}(x_1)\ldots\ldots\ldots \mu_{An}(x_n)) & \text{if } (\exists)\, f^{-1}(y) \\ 0, \text{ otherwise} \end{cases}$$

where x = (x_1, x_2, . . . , x_n)

Explanations

- In the general form, the universe X is the cartesian product of the universes X_1,X_2, . . . X_n, and the fuzzy set A is the cartesian product of the fuzzy sets A_1, A_2, . . . , A_n, i.e. A = $A_1 \times A_2 \times \ldots \times A_n$
- According to the formula for the cartesian product, it results that $\mu_A(x) = \min(\mu_{A1}(x_1)\ldots\ldots\ldots \mu_{An}(x_n))$ where x = (x_1, x_2, . . . , x_n)
- the extension principle in the reduced form can be obtained from the extension principle in the general form, by making n = 1.

X. CONCLUSION

In this paper we have discussed about fuzzy sets and its related phenomena. Since its inception in 1965 as a generalization of classical set theory, it has been advanced to a powerful mathematical theory. It has been applied to many mathematical areas, such as algebra, analysis, clustering, control theory, graph theory, measure theory, optimization, operations research, topology, and so on. In addition, alone or in combination with classical approaches it has been applied in practice in various disciplines, such as control, data processing, decision support, engineering, management, logistics, medicine, and others. It is particularly well suited as a ‘bridge’ between natural language and formal models and for the modeling of nonstochastic uncertainties.

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