3-Equitable Prime Cordial Labeling in Theta Graph

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Abstract: Let $G=(V,E)$ be a simple graph with p vertices and q edges. A 3-equitable prime cordial labeling of a graph G is a bijection f from $V(G)$ to $\{1,2,\ldots|V(G)|\}$ such that if an edge uv is assigned the label 1 if $\gcd(f(u),f(v))=1$ and $\gcd(f(u)+f(v), f(u)-f(v))=1$, the lable 2 if $\gcd(f(u),f(v))=1$ and $\gcd(f(u)+f(v), f(u)-f(v))=2$, and the label 0 otherwise, then the number of edges labeled with i and the number of edges labeled with j differ by at most 1 for $0\le i\le 2$ and $0\le j\le 2$. If a graph has a 3-equitable prime cordial labeling, then it is called a 3-equitable prime cordial graph. In this paper, we discuss 3-equitable prime cordial labeling in the context of some graph operations namely duplication, switching, fusion, path union of two copies and the open star graph of Theta graph .

Keywords: 3-equitable prime cordial labeling, duplication, fusion, switching, path union and open star of graphs

***AMS Mathematics Subject Classification (2010)*:** 05C78

I. Introduction

Graph labeling have enormous applications within mathematics as well as to several areas of computer science and communication networks. In this paper, we consider only finite, simple undirected graphs. For graph theoretic notations and terminology we follow Harary [4] and for number theory we follow Burton [1]. A labeling of a graph G is a mapping that carries vertices and/or edges into a set of numbers, usually integers. In the present work, denotes the Theta graph with 7 vertices and 8 edges. A current survey of various graph labeling problems can be found in Gallian [3]. We shall give a brief summary of results which are useful in the present paper.

Definition 1.1: *A binary vertex labeling* f *of a graph* G *is called a cordial labeling if* $|v_f(0)-v_f(1)| \le 1$ *and* $|e_f(0)-e_f(1)| \le 1$.

A graph which admits cordial labeling is called *a cordial graph*. The concept of cordial labeling was introduced by Cahit [2].

Definition 1.2: *A prime cordial labeling of a graph G with vertex set* $V(G)$ *is a bijection* $f:V(G)\to \{1,2,3,\ldots,|V(G)|\}$ *defined by*

$$f(e=uv)=\begin{cases}1 & if \quad gcd(f(u),f(v))=1;\\ 0 & otherwise\end{cases}$$

further $|e_f(0)-e_f(1)|\le 1$.

A graph which admits prime cordial labeling is called a *prime cordial graph*. The concept of prime cordial labeling was introduced by Sundaram et al. [6].

Definition 1.3: *A* 3*-equitable prime cordial labeling of a graph G with vertex set* $V(G)$ *is a bijection* $f:V(G)\to \{1,2,3,\ldots,|V(G)|\}$ *defined by*

$$f(e=uv)=\begin{cases} 1 \quad if \ \gcd(f(u),f(v))=1 \ and \ \gcd(f(u)+f(v),f(u)-f(v))=1 \\ 2 \quad if \ \gcd(f(u),f(v))=1 \ and \ \gcd(f(u)+f(v),f(u)-f(v))=2 \\ 0 \quad otherwise \end{cases}$$

further $|e_f(i)-e_f(j)|\leq 1$ for all $0\leq i,j\leq 2$.

A graph which admits 3-equitable prime cordial labeling is called a 3-*equitable prime cordial graph*. The concept of 3-equitable prime cordial labeling was introduced by Murugesan et al. [5].

Now let us recall the definition of Theta graph and the graph operations such as duplication, switching, fusion and path union of open star of a graph.

Definition 1.4. *A Theta graph $\theta(\alpha,\beta,\gamma)$ consists of three vertex disjoint paths of length α,β,γ having common end point, where $\alpha\leq\beta\leq\gamma$.*

Throughout this paper, we consider the Theta graph $\theta(2,3,3)$ only and we denote this graph by T_α, we fix the position of vertices in Theta graph T_α as mentioned in the Figure 1.

Figure 1: Theta graph T_α

Definition 1.5. *Duplication of a vertex v_k of a graph G produces a new graph G_1 by adding a vertex v_k' with $N(v_k')=N(v_k)$. In other words, a vertex v_k' is said to be a duplication of v_k if all the vertices which are adjacent to v_k are adjacent to v_k' also.*

Definition 1.6. *A vertex switching of a graph G is a graph G_v obtained from G by removing all the edges incident to v and adding edges joining v to every other vertices which are not adjacent to v in G.*

Definition 1.7. *Let u and v be any two distinct vertices of a graph G. A new graph G_1 is constructed by fusing (identifying) two vertices u and v by a single vertex x in G_1 such that every edge which was incident with either u or v in G is now incident with x in G_1.*

Definition 1.8. *Let $G_1,G_2,G_3,\ldots G_n$, $n\geq 2$ be n copies of a fixed graph G. The graph obtained by adding an edge between G_i and G_i+1 for $i=1,2,\ldots n-1$ is called the path union of G.*

Definition 1.9. *A graph obtained by replacing each vertex of $K_{1,n}$ except the apex vertex by the graphs $G_1, G_2,\ldots,G_n$ is known as open star of graphs. We shall denote such graph by $S(G_1, G_2,\ldots,G_n)$.*

If we replace each vertices of $K_{1,n}$ except the apex vertex by a graph G. i.e. $G_1=G_2=\cdots=G_n=G$, such open star of a graph, is denoted by $S(n\cdot G)$.

2. Main results:

Theorem 2.1. *The Theta graph T_α is a 3-equitable prime cordial graph.*

Proof: Let $v_0, v_1, v_2, \ldots, v_6$ are the vertices of the Theta graph T_α with v_0 be the central vertex and $E(T_\alpha) = \{v_i v_{i+1} : 1 \le i \le 5\} \cup \{v_0 v_1, v_0 v_4, v_1 v_6\}$, then $|V(T_\alpha)| = 7$ and $|E(T_\alpha)| = 8$.

We define vertex labeling $f : V(T_\alpha) \to \{1,2,3,\ldots,7\}$ as follows.

$f(v_0) = 6, f(v_1) = 3, f(v_2) = 7, f(v_3) = 4, f(v_4) = 2, f(v_5) = 5, f(v_6) = 1$

For the graph T_α the possible pairs of labels of adjacent vertices are

$(1,2),(1,3),(1,4),(1,5),(1,6),(1,7), (2,3),(2,4),(2,5),(2,6),(2,7), (3,4),(3,5),(3,6),(3,7), (4,5),(4,6),(4,7), (5,6),(5,7),(6,7)$.

Then $|e_f(i) - e_f(j)| \le 1$ for $i, j = 0, 1, 2.$

Therefore, T_α is a 3-equitable prime cordial graph.

Figure 2: 3-equitable prime cordial Theta graph T_α

Theorem 2.2. *The duplication of any vertex in the Theta graph T_α is a 3-equitable prime cordial graph.*

Proof: Let $v_0, v_1, v_2, \ldots, v_6$ are the vertices of the Theta graph T_α with centre v_0 and $E(T_\alpha) = \{v_i v_{i+1} : 1 \le i \le 5\} \cup \{v_0 v_1, v_0 v_4, v_1 v_6\}$, then $|V(T_\alpha)| = 7$ and $|E(T_\alpha)| = 8$.

Let G_i be a graph obtained from T_α after duplication vertex of the vertex v_i in T_α and v_i' be the duplication vertex of the vertex v_i. Clearly $|V(G_i)| = 8$.

We define vertex labeling $f : V(G_i) \to \{1,2,3,\ldots,8\}$ as in the following cases:

Case (i): Duplication of the vertex v_i for $i = 0,1,2,3,5,6.$

We define $f(v_i') = 8$ for $i = 0,1,2,3,5,6.$ where v_i' is the duplicating vertex of v_i.

Further, $f(v_0) = 6,\ f(v_1) = 3, f(v_2) = 7, f(v_3) = 4, f(v_4) = 2, f(v_5) = 5, f(v_6) = 1.$

Case (ii): Duplication of the vertex v_4

We define $f(v_4') = 8$ for $i = 4$ where v_4' is the duplicating vertex of v_4.

Further, $f(v_0) = 2,\ f(v_1) = 3, f(v_2) = 7, f(v_3) = 4, f(v_4) = 6, f(v_5) = 5, f(v_6) = 1.$

Thus in both cases, we have $|e_f(i) - e_f(j)| \le 1$ for $i, j = 0, 1, 2.$

Hence the graph obtained by the duplication of any vertex v_i in the Theta graph T_α is a 3-equitable prime cordial graph.

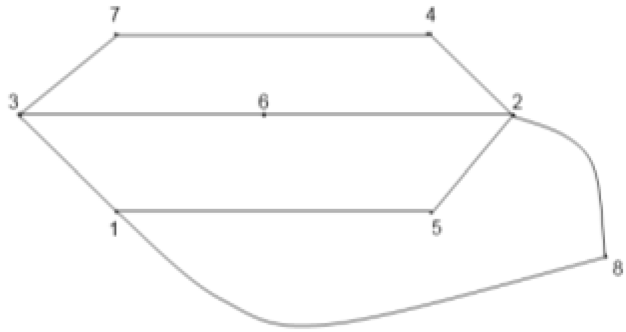

Figure 3: The duplication of the vertex v_5 in T_α is a 3-equitable prime cordial graph

Theorem 2.3. *The switching of any vertex in the Theta graph T_α is a 3-equitable prime cordial graph.*

Proof: If $v_0, v_1, v_2, \ldots, v_6$ are the vertices of the Theta graph T_α with centre v_0 and $E(T_\alpha) = \{v_i v_{i+1} : 1 \le i \le 5\} \cup \{v_0v_1, v_0v_4, v_1v_6\}$, then $|V(T_\alpha)| = 7$ and $|E(T_\alpha)| = 8$.

Let G_s be the graph obtained from T_α after switching the vertex v_i. In T_α, consider only three vertices be v_2, v_3 and v_5. Clearly $|V(G_s)| = 7$.

We define vertex labeling $f : V(G_s) \to \{1,2,3,\ldots,7\}$ as follows.

Case (i): switching of the vertex v_0

We define, $f(v_0) = 6,\ f(v_1) = 2, f(v_2) = 4, f(v_3) = 3, f(v_4) = 7, f(v_5) = 5, f(v_6) = 1.$

Case (ii): switching of the vertex v_3

We define, $f(v_0) = 6,\ f(v_1) = 3, f(v_2) = 7, f(v_3) = 4, f(v_4) = 2, f(v_5) = 5, f(v_6) = 1.$

Case (iii): switching of the vertex v_4

We define, $f(v_0) = 5,\ f(v_1) = 4, f(v_2) = 6, f(v_3) = 2, f(v_4) = 3, f(v_5) = 1, f(v_6) = 7.$

Case (iv): switching of the vertex v_5

We define, $f(v_0) = 6,\ f(v_1) = 3, f(v_2) = 7, f(v_3) = 4, f(v_4) = 2, f(v_5) = 5, f(v_6) = 1.$

Thus in both cases, we have $|e_f(i) - e_f(j)| \le 1$ for $i, j = 0, 1, 2.$

Hence the graph G_s admits 3-equitable prime cordial graph.

Note that switching of vertices v_3, v_4 and v_5 are as similar as the switching of vertices v_2, v_1 and v_6 respectively.

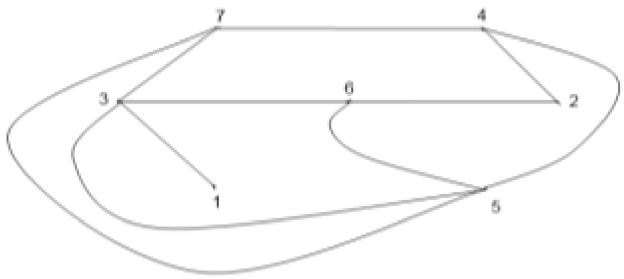

Figure 4: The switching of the vertex v_5 in T_α is a 3-equitable prime cordial graph

Theorem 2.4. *The fusion of any two vertices in the Theta graph T_α is a 3-equitable prime cordial graph.*

Proof: If $v_0, v_1, v_2, \ldots, v_6$ be the vertices of the Theta graph T_α with centre v_0

$E(T_\alpha) = \{v_i v_{i+1} : 0 \le i \le 6\}$, then $|V(T_\alpha)| = 7$ and $|E(T_\alpha)| = 8$.

Let G be a graph obtained by fusion of any two vertices in T_α. Then $|V(G)| = 6$ and $|E(G)| = 7$. We define vertex labeling $f : V(G) \to \{1,2,3,\ldots,|V(G)|\}$ as follows.

For the graph G the possible pairs of labels of adjacent vertices are $(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)$

Out of these pairs, only the pairs $(2,4),(3,6)$ yields the edge label value as 0, the pairs $(1,6),(2,3),(4,5),(5,6)$ yields the edge label value as 1 and the remaining possible labeling of pairs $(1,3),(1,5),(3,5)$ yields the edge label value as 2.

In view of the labeling pattern defined above we have $e_f(0) = 2, e_f(1) = 3, e_f(2) = 2$.

Then $|e_f(i) - e_f(j)| \le 1$ for $i, j = 0,1,2$.

Hence the graph G admits 3-equitable prime cordial graph.

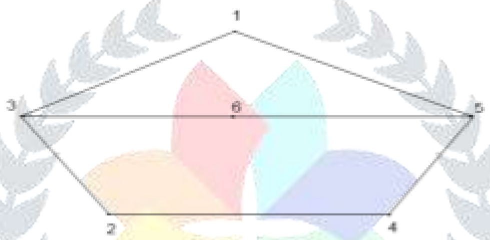

Figure 5: The fusion of the vertex v_2 and v_3 in T_α is a 3-equitable prime cordial graph

Therom 2.5. *The graph G obtained by path union of two copies of theta graph T_α is a 3-equitable prime cordial graph.*

Proof: Let G be the graph obtained by path union of two copies of Theta graph T_α and T_α' respectively. Let $u_0, u_1 \ldots, u_6$ be the vertices of first copy T_α and $v_0, v_1, \ldots, v_6$ be the vertices of second copy T_α'. Note that $V(G) = V(T_\alpha) \cup V(T_\alpha')$ and $E(G) = E(T_\alpha) \cup E(T_\alpha') \cup \{u_k v_k\}$.

Then $|V(G)| = 14$ and $|E(G)| = 17$.

We define vertex labeling $f : V(G_s) \to \{1,2,3,\ldots,|V(G)|\}$ such that

$f(u_0) = 3, f(u_1) = 1, f(u_2) = 2, f(u_3) = 4, f(u_4) = 5, f(u_5) = 10, f(u_6) = 12$

and $f(v_0) = 9, f(v_1) = 7, f(v_2) = 6, f(v_3) = 8, f(v_4) = 11, f(v_5) = 13, f(v_6) = 14$.

In view of the labeling pattern defined above we have $e_f(0) = 6, e_f(1) = 6, e_f(2) = 5$.

Thus we have $|e_f(i) - e_f(j)| \le 1$ for $i, j = 0,1,2$.

Hence G is a 3-equitable prime cordial graph.

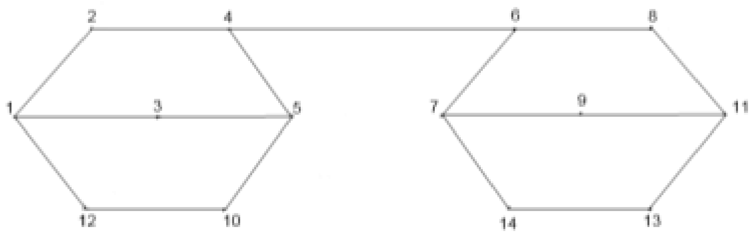

Figure 6: The path union of two copies of T_α is a 3-equitable prime cordial graph

Theorem 2.6. $S(n \cdot T_\alpha)$ *is a 3-equitable prime cordial graph, where n is even.*

Proof: Let *G* be a graph obtained by replacing each vertices of $K_{1,n}$ except the central vertex by the Theta graph T_α , where *n* is any positive integer, i.e. $G = S(n \cdot T_\alpha)$.

We fix the position of the vertices in each copies of Theta graph as follows:

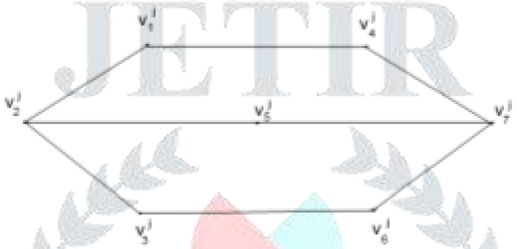

Figure 7: Theta graph

Let u_0 be the central vertex of the graph *G*, where u_0 is an apex vertex of the orginal graph $K_{1,n}$.

We denote u_i^j $(or\ v_i^j)$ is the *i*th vertex in the *j*th copy of T_α , where $1 \le i \le 7; 1 \le j \le n$. Then $V(G) = \left\{ u_i^j, v_i^k;\ 1 \le i \le 7,\ 1 \le j \le \frac{n}{2},\ \frac{n}{2} + 1 \le k \le n \right\}$

Now we shall join each *i*th vertex of T_α to the apex vertex u_0 , where *i* is any fixed integer between 1 and 7. Then $|V(G)| = 7n + 1$ and $|E(G)| = 9n$.

We shall define the labeling function $f : |V(G)| \to \{1, 2, ..., 7n + 1\}$ as follows:

Let $f(u_0) = 1.$

Case 1: when $j = 1, 2, ..., \frac{n}{2}$

$$f(u_i^j) = \begin{cases} 2i + 1 + 14(j - 1)\ ;\ 1 \le i \le 4 \\ 2i - 8 + 14(j - 1)\ ;\ 5 \le i \le 7 \end{cases}$$

Case 2: when $j = \frac{n}{2} + 1, ..., n$

$$f(v_i^j)=\begin{cases} 11+2(i-1)+14(j-\frac{n}{2}-1)\ ;\ 1\le i\le 3 \\ 2i+14(j-\frac{n}{2}-1)\ ;\ 4\le i\le 7 \end{cases}$$

Thus we have $|e_f(i)-e_f(j)|\le 1$ for $i, j=0,1,2.$

Hence the above labeling pattern gives 3-equitable prime cordial labeling to the grpah G and so it is a 3-equitable prime cordial graph.

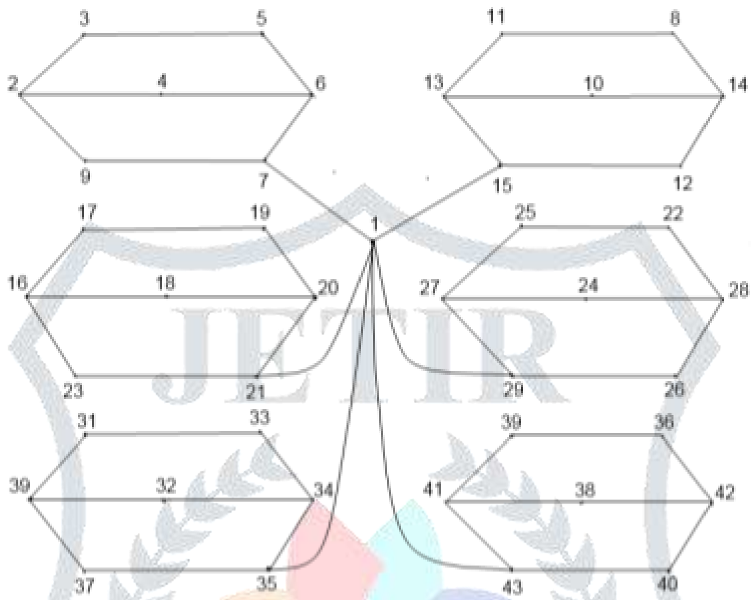

Figure 8: A open star of 6 copies of T_α and its prime cordial labeling.

3. Concluding remarks:

In this paper, we investigated the 3-equitable prime cordial labeling of theta graph. We also proved that the 3-equitable prime cordial labeling in the context of some graph operations namely duplication, switching, fusion, path union of two copies of Theta graph and the open star of Theta graph. However we proved that the path union of only two copies of Theta graph is a 3-equitable prime cordial graph. An interesting open area of research is to extend the arbitrary number of path union of Theta graphs.

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