INVERSE (G, D)-NUMBER OF A GRAPH

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Abstract: *Let S be a minimum (G, D)-set in a graph G. If $V - S$ contains a (G, D)-set S^{-1} of G, then S^{-1} is called an inverse (G, D)-set with respect to S. If $V - S$ has no (G, D)-set, then inverse (G, D)-set does not exist in G. An inverse (G, D)-set S is called a minimum inverse (G, D)-set, if S consists of minimum number of vertices among all inverse (G, D)-sets. The number of vertices in a minimum inverse (G, D)-set is defined as the inverse (G, D)-number of a graph G, and is denoted by ${\gamma_G}^{-1}(G)$. If G has no inverse (G, D)-set, then it is defined as ${\gamma_G}^{-1}(G) = \infty$. In this paper, we initiate the study of this parameter.*
Keywords: *Domination, Geodomination, (G, D)-set, Inverse (G, D)-set and Inverse (G, D)- number.*

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1. Introduction: Throughout this paper, we consider the graph G as a finite undirected simple graph with no loops and multiple edges. The study of domination in graphs was begun by Ore and Berge[5]. Let G = (V, E) be any graph. A dominating set of a graph G is a set D of vertices of G such that every vertex in V–D is adjacent to at least one vertex in D and the minimum cardinality among all dominating sets is called the domination number γ(G) of G. The concept of geodominating(or geodetic) set was introduced by Buckley and Harary in [1] and Chartrand, Zhang and Harary in [2], [3], [4]. Let u, v ∈ V(G). A u-v geodesic is a u-v path of length d(u, v). A vertex x ∈ V(G) is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. A set S of vertices of G is a geodominating(or geodetic) set if every vertex of G lie on an x-y geodesic for some x, y in S. The minimum cardinality of a geodominating set is the geodomination(or geodetic) number of G and is denoted as g(G)[1], [2], [3], [4]. A (G, D)-set of G is a subset S of V(G) which is both a dominating and geodetic set of G. A (G, D)-set S of G is said to be a minimal (G, D)-set of G if no proper subset of S is a (G, D)-set of G. The minimum cardinality of all (G, D)-sets of G is called the (G, D)-number of G and it is denoted by γ_G(G). Any (G, D)-set of G of cardinality γ_G is called a γ_G-set of G [8, 9, 10]. A γ_G-required vertex is a vertex which lie in every (G, D)-set of G.

The concept of inverse domination was introduced by Kulli in [7]. Let D be a γ-set of G. A dominating set D' contained in $V - D$ is called an inverse dominating set of G with respect to D. Motivated by this definition, in this paper, we introduce the new parameter inverse (G,D)-number, investigate its properties and find its value for some standard graphs.

Proposition 1.1:[8] For $n > 5$, $\gamma_G(C_n) = \gamma(C_n) = \left\lceil\frac{n}{3}\right\rceil$.

Remark 1.2:[6] (G, D)-number of a disconnected graph is the sum of (G, D)-number of its components.

2. Main Results

Definition 2.1: Let S be a minimum (G, D)-set in a graph G. If $V - S$ contains a (G, D)-set S^{-1} of G, then S^{-1} is called an inverse (G, D)-set with respect to S. If $V - S$ has no (G, D)-set, then inverse (G, D)-set does not exist in G.

Definition 2.2: An inverse (G, D)-set S is called a minimum inverse (G, D)-set, if S consists of minimum number of vertices among all inverse (G, D)-sets.

Definition 2.3: The number of vertices in a minimum inverse (G, D)-set is defined as the inverse (G, D)-number of a graph G, and is denoted by $\gamma_G{}^{-1}(G)$. If G has no inverse (G, D)-set, then it is defined as $\gamma_G{}^{-1}(G) = \infty$.
Observation 2.4: If a graph G has isolated vertex, pendant vertex or extreme vertex, then $\gamma_G{}^{-1}(G) = \infty$.

Example 2.5: (i) $\gamma_G{}^{-1}(P_n) = \infty$. (ii) Consider the graph G as in figure (2.1).

G:

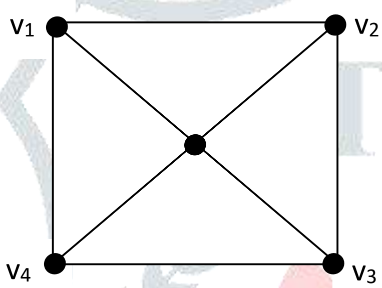

Figure (2.1)

In G, $S_1 = \{v_1, v_3\}$ and $S_2 = \{v_2, v_4\}$ are minimum (G, D)-sets. Their corresponding inverse (G, D)-sets are $S_1{}^{-1} = \{v_2, v_4\}$ and $S_2{}^{-1} = \{v_1, v_3\}$ with respect to $S_1$and S_2 respectively. Therefore, $\gamma_G(G) = 2$ and $\gamma_G{}^{-1}(G) = 2$.

Proposition 2.6: A graph G in which $\gamma_G{}^{-1}(G)$ exists if and only if G has no γ_G-required vertex.

Proof: Let G be a graph with $\gamma_G{}^{-1}(G)$ exist. Let S be a γ_G-set and S^{-1} be a $\gamma_G{}^{-1}$-set of G. Suppose G contains a γ_G-required vertex u. Then, u lies in every γ_G-set of G and hence, $u \in S$ and $u \in S^{-1}$. Which is in contradiction to $S^{-1} \subseteq V - S$.
Conversely, if G has a γ_G-required vertex. Then, obviously, $V - S$ does not contain any (G, D)-set for all minimum (G, D)-set S of G. Therefore, inverse (G, D)-set does not exists. Hence, $\gamma_G{}^{-1}(G) = \infty$.

Proposition 2.7: Given a positive integer $k \geq 2$, there exist a graph G with $\gamma_G{}^{-1}(G) = k$.

Proof: Graph G is given in the following figure (2.2).
Clearly, $S = \{a, b\}$ is a γ_G-set of G and so $\gamma_G(G) = 2$. Here, $V(G) - S = \{v_i : 1 \leq i \leq k\}$. For $k \geq 2$, $V(G) - S$ is a (G, D)-set of G with respect to S. Thus, $V(G) - S$ is a $\gamma_G{}^{-1}$-set of G.
Therefore, $\gamma_G{}^{-1}(G) = |V(G) - S| = k$.

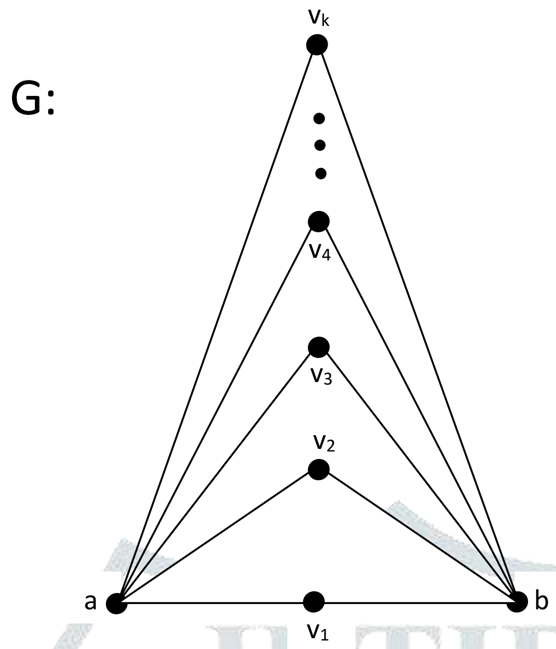

Figure (2.2)

Observation 2.8: (i) For any graph G, $\gamma_G(G) \leq \gamma_G{}^{-1}(G)$.
(ii) For any graph G, $4 \leq \gamma_G(G) + \gamma_G{}^{-1}(G) \leq n$.
Lower bound holds for C_4 and upper bound holds for the graph in figure (2.2).
(iii) $\gamma_G{}^{-1}(C_3) = \gamma_G{}^{-1}(C_5) = \infty$.

Proposition 2.9: $\gamma_G{}^{-1}(W_n) = \begin{cases} \frac{n-1}{2} & if\ n\ is\ odd \\ \infty & otherwise. \end{cases}$

Proof: Let $W_n \cong C_{n-1} + K_1$.
Case 1: n is odd
Since n is odd, $n-1$ is even. Let S be a γ_G-set of W_n. Then, $S^{-1} = \{u_{i+1}: u_i \in S\}$ or $\{u_{i-1}: u_i \in S\}$ is a minimum inverse (G,D)-set of W_n. Therefore, $\gamma_G{}^{-1}(W_n) = \frac{n-1}{2}$.
Case 2: n is even
In this case, W_n contains exactly one (G,D)-set. Thus, $\gamma_G{}^{-1}(W_n)$ does not exists and so, $\gamma_G{}^{-1}(W_n) = \infty$.

Proposition 2.10: For $n > 5$, $\gamma_G{}^{-1}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proof: Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1}: 1 \leq i \leq n-1\} \cup \{v_n v_1\}$.
When $n \equiv 0(mod\ 3)$, C_n contains exactly three disjoint (G,D)-sets of same order.
When $n \not\equiv 0(mod\ 3)$, C_n contains exactly two disjoint (G,D)-sets and their cardinalities are same. Therefore, $\gamma_G{}^{-1}(C_n)$ exist and hence, $\gamma_G{}^{-1}(C_n) = \gamma_G(C_n) = \left\lceil \frac{n}{3} \right\rceil$ (By Proposition 1.1).
Remark 2.11: Let $G_1, G_2, \ldots, G_k$ be the k components of a graph G. Let $S_1, S_2, \ldots, S_k$ be γ_G-sets and $S_1{}^{-1}, S_2{}^{-1}, \ldots, S_k{}^{-1}$ be $\gamma_G{}^{-1}$-sets of $G_1, G_2, \ldots, G_k$ with respect to $S_1, S_2, \ldots, S_k$ respectively. Then, $S_1{}^{-1} \cup S_2{}^{-1} \cup \ldots \cup S_k{}^{-1}$ is a $\gamma_G{}^{-1}$-set of G. Therefore, $\gamma_G{}^{-1}(G) = \sum_{i=1}^{k} \gamma_G{}^{-1}(G_i)$.

Theorem 2.12: Let $G_1, G_2, \ldots, G_k$ be the k components of a graph G. Then, $\gamma_G(G) = \gamma_G{}^{-1}(G)$ if and only if $\gamma_G(G_i) = \gamma_G{}^{-1}(G_i)$ for $i = 1$ to k.

Proof: Let $G_1, G_2, \ldots, G_k$ be the k components of G. Then, by Remark (1.2), $\gamma_G(G) = \sum_{i=1}^{k} \gamma_G(G_i)$.

By Remark (2.11), $\gamma_G{}^{-1}(G) = \sum_{i=1}^{k} \gamma_G{}^{-1}(G_i)$.
Thus, trivially, $\gamma_G(G) = \gamma_G{}^{-1}(G)$ if $\gamma_G(G_i) = \gamma_G{}^{-1}(G_i)$ for $i = 1$ to k.
Conversely, assume that $\gamma_G(G) = \gamma_G{}^{-1}(G)$. We have, $\gamma_G(G_i) \leq \gamma_G{}^{-1}(G_i)$ for $i = 1$ to k.
Suppose, $\gamma_G(G_i) < \gamma_G{}^{-1}(G_i)$ for some i.
Then, we must have $\gamma_G(G_j) > \gamma_G{}^{-1}(G_j)$ for some j$\neq i$. Which is impossible.
Hence, $\gamma_G(G_i) = \gamma_G{}^{-1}(G_i)$ for $i = 1$ to k.

Theorem 2.13: $\gamma_G{}^{-1}(G_1 \cup G_2) = \gamma_G{}^{-1}(G_1) + \gamma_G{}^{-1}(G_2)$.

Proof: The proof follows from the fact that $\gamma_G{}^{-1}$-set of $G_1 \cup G_2$ is the union of $\gamma_G{}^{-1}$-set of G_1 and $\gamma_G{}^{-1}$-set of G_2.

Theorem 2.14: $\gamma_G{}^{-1}$(Cube) = 2.

Proof: Cube is a 3-regular graph with 8 vertices. V(Cube) = $\{v_i: 1 \leq i \leq 8\}$.

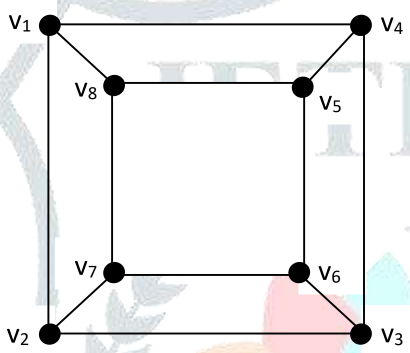

Figure (2.3)

Let $S = \{v_1, v_6\}$. Then, S dominate and geodominate all the vertices of $V - S$. So, S is a γ_G-set.
Therefore, γ_G(Cube) = 2. Now, $V - S = \{v_2, v_3, v_4, v_5, v_7, v_8\}$. Clearly, $S^{-1} = \{v_2, v_5\}$ is a minimum inverse (G, D)-set of Cube with respect to S. Hence, $\gamma_G{}^{-1}$(Cube) = $|S^{-1}| = 2$.

Theorem 2.15: $\gamma_G{}^{-1}$(Octahedron) = 2.

Proof: Octahedron is a 4-regular graph with 6 vertices. V(Octahedron) = $\{v_i: 1 \leq i \leq 6\}$.

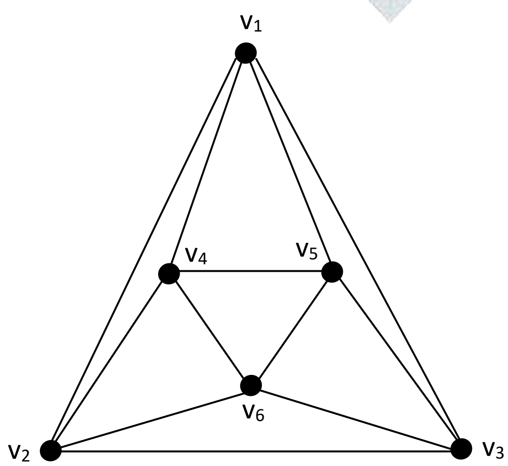

Figure (2.4)

Let $S = \{v_1, v_6\}$. Then, S dominate and geodominate all the vertices of $V - S$. So, S is a γ_G-set. Therefore, γ_G(Octahedron) = 2. Now, $V - S = \{v_2, v_3, v_4, v_5\}$. So, $S^{-1} = \{v_2, v_5\}$ is a minimum inverse (G, D)-set of Octahedron with respect to S. Hence, ${\gamma_G}^{-1}$(Octahedron) = $|S^{-1}| = 2$.

Theorem 2.16: ${\gamma_G}^{-1}$(Icosahedron) = 2.

Proof: Icosahedron is a 5-regular graph with 12 vertices. V(Icosahedron) = $\{v_i : 1 \leq i \leq 12\}$.

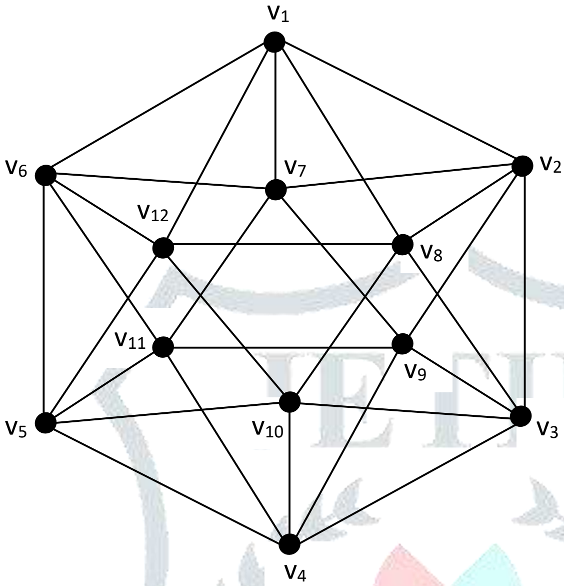

Figure (2.5)

Let $S = \{v_1, v_4\}$. Then, S dominate and geodominate all the vertices of $V - S$. So, S is a γ_G-set. Therefore, γ_G(Icosahedron) = 2. Now, $V - S = \{v_2, v_3, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$. So, $S^{-1} = \{v_7, v_{10}\}$ is a minimum inverse (G, D)-set of Icosahedron with respect to S. Hence, ${\gamma_G}^{-1}$(Icosahedron) = $|S^{-1}| = 2$.

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