Applicability and Capability of Generalized Gamma Model using R Software

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Abstract: In this paper, we have considered the Generalized Gamma Distribution with its structural properties. Various special cases of proposed model are defined and studied. The estimation of parameters of said model and its special cases are obtained by employing the computational MLE technique. A comparative study has been done between Generalized Gamma Distribution and its special cases by using -log L, AIC and BIC criteria on the basis of simulated and real life data sets. From this study, it was observed that Generalized Gamma, Gamma and Rayleigh Distributions are better fitted models as compared to other probabilistic models for simulated as well as proposed real life data sets.

Key words: Generalized Gamma Distribution, MLE, AIC, BIC and R software.

I. INTRODUCTION

The generalized gamma (GG) distribution presents a flexible family in the varieties of shapes and hazard functions for modeling duration. It was introduced by Stacy (1962). The GG family, which encompasses exponential, gamma, and Weibull as subfamilies, and lognormal as a limiting distribution, has been used in economics by Jaggia (1991). Prentice (1974) resolved the convergence problem using a nonlinear transformation of GG model. However, despite its long history and growing use in various applications, the GG family and its properties has been remarkably presented in different papers..Hwang, T et al (2006) introduced a new moment estimation of parameters of the generalized gamma distribution using its characterization. In information theory, thus far a maximum entropy (ME) derivation of GG is found in Kapur (1989) where it is referred to as generalized Weibull distribution and the entropy of GG has appeared in the context of flexible families of distributions. Other contribution of GG distribution can be found in the literature (see Reshi et al (2014),Kaisar et al (2015)). The probability density function of the generalized gamma distribution (GG (α, **τ**, **λ**)) is given by

$$f(x;\tau,\alpha,\lambda)=\frac{\tau}{\lambda\Gamma(\alpha)}\left(\frac{x}{\lambda}\right)^{\alpha\tau-1}e^{-\left(x/\lambda\right)^{\tau}};x\geq 0,\ \tau,\ \alpha,\ \lambda>0 \qquad (1.1)$$

With cumulative distribution function of $X \sim GG(\alpha,\tau,\lambda)$ as:

$$F(x)=\frac{\Gamma\left(\alpha,\left(\frac{x}{\lambda}\right)^{\tau}\right)}{\Gamma(\alpha)} \qquad (1.2)$$

Where $\Gamma(.)$ is the gamma function, α and **τ** are shape parameters and **λ** is the scale parameter. The GG family is flexible in that it includes several well-known models as sub families the subfamilies of GG thus far considered in the literature are exponential (α=**τ**=1),gamma for (**τ**=1),and Weibull for (α=1).The lognormal distribution is also obtained as a limiting distribution when $\alpha \to \infty$. By letting **τ**=2 we obtain a subfamily of GG which is known as the Generalized normal distribution (GN).The GN is itself a flexible family and includes half –normal $\left(\alpha=\frac{1}{2},\lambda^2=2\sigma^2\right)$,Rayleigh $\left(\alpha=1,\lambda^2=2\sigma^2\right)$, Maxwell-Boltzmann $\left(\alpha=\frac{3}{2}\right)$and chi $\left(\alpha=\frac{k}{2},k=1,2,...\right)$.

The survival function, Hazard Function and reverse hazard rate function are given by

$$S(x)=1-\frac{\Gamma\left(\alpha;\left(\frac{x}{\lambda}\right)^{\tau}\right)}{\Gamma(\alpha)} \qquad (1.3)$$

$$\lambda(x)=\frac{\frac{\tau}{\lambda\Gamma(\alpha)}\left(\frac{x}{\lambda}\right)^{\alpha\tau-1}e^{-\left(\frac{x}{\lambda}\right)^{\tau}}}{1-\frac{\Gamma\left(\alpha\left(\frac{x}{\lambda}\right)^{\tau}\right)}{\Gamma(\alpha)}} \qquad (1.4)$$

$$RHR = \frac{\frac{\tau}{\lambda\Gamma(\alpha)}\left(\frac{x}{\lambda}\right)^{\alpha\tau-1} e^{-\left(\frac{x}{\lambda}\right)^{\tau}}}{\frac{\Gamma\left(\alpha,\left(\frac{x}{\lambda}\right)^{\tau}\right)}{\Gamma(\alpha)}} \qquad (1.5)$$

2. Structural Properties of Generalized Gamma Distribution:

The summary Statistics of GGD are given by:

$$\mu_1' = \frac{\lambda}{\Gamma(\alpha)}\Gamma\left(\frac{1}{\tau}+\alpha\right) \qquad (2.1)$$

$$\mu_2 = \frac{\lambda^2}{\Gamma(\alpha)}\Gamma\left(\frac{2}{\tau}+\alpha\right) - \left(\frac{\lambda}{(\alpha)}\Gamma\left(\frac{1}{\tau}+\alpha\right)\right)^2 \qquad (2.2)$$

$$\gamma_1 = \frac{\frac{\lambda^3}{\Gamma(\alpha)}\left(\Gamma\left(\alpha+\frac{3}{\tau}\right) - \frac{3}{\Gamma(\alpha)}\left(\Gamma\left(\alpha+\frac{2}{\tau}\right)\right)\left(\Gamma\left(\alpha+\frac{1}{\tau}\right)\right) + \frac{2}{(\Gamma(\alpha))^2}\left(\Gamma\left(\alpha+\frac{1}{\tau}\right)\right)\right)}{\left(\frac{\lambda^2}{\Gamma(\alpha)}\left(\Gamma\left(\alpha+\frac{2}{\tau}\right) - \frac{\Gamma\left(\alpha+\frac{1}{\tau}\right)}{\Gamma(\alpha)}\right)^3\right)^{1/2}} \qquad (2.3)$$

$$\gamma_2 = \frac{\Gamma(\alpha)\left(\Gamma\left(\alpha+\frac{4}{\tau}\right) - \frac{4}{\Gamma(\alpha)}\left(\Gamma\left(\alpha+\frac{3}{\tau}\right)\right)\left(\Gamma\left(\alpha+\frac{1}{\tau}\right)\right) + \frac{6}{\Gamma(\alpha)}\left(\Gamma\left(\alpha+\frac{2}{\tau}\right)\right)\left(\Gamma\left(\varepsilon+\frac{1}{\tau}\right)\right) - \frac{3}{(\Gamma(\alpha))^3}\left(\Gamma\left(\varepsilon+\frac{1}{\tau}\right)\right)^4\right)}{\left(\Gamma\left(\varepsilon+\frac{2}{\tau}\right) - \frac{\left(\Gamma\left(\alpha+\frac{1}{\tau}\right)\right)^2}{\Gamma(\alpha)}\right)^2} - 3 \qquad (2.4)$$

$$\text{C. V} = \frac{\left(\frac{\lambda^2}{\Gamma(\alpha)}\Gamma\left(\alpha+\frac{2}{\tau}\right) - \frac{\lambda}{\Gamma(\alpha)}\Gamma\left(\alpha+\frac{1}{\tau}\right)\right)^{1/2}}{\frac{\lambda}{\Gamma(\alpha)}\Gamma\left(\alpha+\frac{1}{\tau}\right)} \qquad (2.5)$$

3. Information Measures of GGD:

The concept of Shannon's entropy is a central role of information theory, sometimes referred to as measure of uncertainty. The entropy of a random variable is defined in terms of its probability distribution and can b shown as a good measure of randomness or uncertainty. Henceforth we assume that log is to the base 2 and entropy is expressed in terms of bits. The Shannon entropy of GGD is given as:

$$\begin{aligned} H(GG) &= -\log\tau + \alpha\tau\log\lambda + \log\Gamma(\alpha) - (\alpha\tau-1)\left(\tfrac{\Psi(\alpha)}{\tau} + \log\lambda\right) + \alpha \\ &= \log\lambda + \log\Gamma(\alpha) + \alpha - \log\tau + \left(\tfrac{1}{\tau} - \alpha\right)\Psi(\alpha) \end{aligned} \qquad (3.1)$$

We can summarize the entropy of subfamilies of GG distribution as below table.

Distribution	**Parameters**			**Entropy**
	α	τ	λ	
Exponential	1	1	λ	$\log\lambda + 1$
Gamma	α	1	λ	$\log\lambda + \log\Gamma(\alpha) + \alpha + (1-\alpha)\Psi(\alpha)$
Weibull	1	τ	λ	$\log\lambda + 1 - \log\tau + \left(\frac{1}{\tau} - 1\right)\Psi(1)$

Generalized normal	α	2	λ	$\log\lambda+\log\Gamma(\alpha)+\alpha-1+\left(\frac{1}{2}-\alpha\right)\Psi(\alpha)$
Half normal	0.5	2	$\sqrt{2\sigma^2}$	$\log\sigma+\log\sqrt{\pi}$
Rayleigh	1	2	$\sqrt{2\sigma^2}$	$\frac{1}{2}+\log\sigma-\frac{1}{2}\Psi(1)$
Maxwell Boltzmann	$\frac{3}{2}$	2	λ	$\log\lambda+\log\frac{\sqrt{\pi}}{2}-\frac{1}{2}-\Psi\left(\frac{3}{2}\right)$
Chi	$\frac{k}{2}$	2	λ	$\log\lambda+\log\Gamma\left(\frac{k}{2}\right)+\frac{k-2}{2}+\left(\frac{1-k}{2}\right)\Psi\left(\frac{k}{2}\right)$

4. Maximum Likelihood Estimation (MLE)

Let $x_{1,}x_2,\dots,x_n$ be a random sample of size n drawn from the probability density function

$$f(x;\alpha,\tau,\lambda)=\frac{\tau}{\lambda\Gamma(\alpha)}\left(\frac{x}{\lambda}\right)^{\alpha\tau-1}e^{-\left(\frac{x}{\lambda}\right)^{\tau}}\ ;x\geq 0,\ \alpha,\tau,\lambda>0$$

The likelihood function is given by

$$L=\left(\frac{\tau}{\lambda\Gamma(\alpha)}\right)^{n}\prod_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\alpha\tau-1}e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\tau}} \tag{4.1}$$

and the logarithm of the likelihood is given by

$$\log L=n\log\tau-n\alpha\tau\log\lambda-n\log\Gamma(\alpha)+(\alpha\tau-1)\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\tau}$$

The first order conditions for finding the optimal values of the parameters are obtained by differentiating with respect to τ,α and λ we get the following differential equations

$$\frac{\delta\log L}{\delta\tau}=\frac{n}{\tau}-n\alpha\log\lambda+\alpha\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\left(\frac{x}{\lambda}\right)^{\tau}\log\left(\frac{x_i}{\lambda}\right) \tag{4.2}$$

$$\frac{\delta\log L}{\delta\alpha}=n\Psi(\alpha)+\tau\sum_{i=1}^{n}\log x_i-n\tau\log\lambda \tag{4.3}$$

$$\frac{\delta\log L}{\delta\lambda}=-\frac{n\alpha\tau}{\lambda}-+\left(\frac{\tau}{\lambda}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\tau} \tag{4.4}$$

These three derivative equations can be solved by Newton –Raphson method. The Newton-Raphson method is a powerful technique for solving equations numerically. In practice $\hat{\tau},\hat{\alpha}$ and $\hat{\lambda}$ are the solutions of the estimating equations obtained by differentiating the likelihood in terms of τ,α and λ solving in (a) to (b) to zero. Therefore, $\hat{\tau},\hat{\alpha}$ and $\hat{\lambda}$ can be obtained by solving the resulting equations simultaneously using a numerical procedure with Newton-Raphson method.

5. Applicability and capability of Generalized Gamma Distribution:

Here we analyze the strength data, reported by Badar and Priest (1982), using Generalized Gamma Distribution. Estimates of the unknown parameters, and hence of R, are obtained by both the methods discussed. It may be noted that Raqab et al. (2008) fitted the 3-reported strength data measured in GPA for single carbon fibre and impregnated 1000 carbon fibre tows. Single fibre were tested at gauge lengths of 1, 10, 20, 50 mm. Impregnated tows of 1000 fibres at gauge lengths of 20, 50, 150 and 300mm. Data set I of size 63 correspond to single fibre with 20 mm of gauge length.

Data set 1:0.101, 0.332, 0.403, 0.428, 0.457, 0.550, 0.561, 0.596, 0.597, 0.645, 0.954, 0.674, 0.718, 0.722, 0.725, 0.732, 0.775, 0.814, 0.816, 0.818, 0.824, 0.859, 0.875, 0.938, 0.940, 1. 056, 1.117, 1.128, 1.137, 1.137, 1.177, 1.196, 1.230, 1.325, 1.339, 1.345, 1.420, 1.423, 1.435, 1.443, 1.464, 1.472, 1.494, 1.532, 1.546, 1.577, 1.608, 1.635, 1.693, 1.701, 1.737, 1.754, 1.762, 1.828, 2. 052, 2. 071, 2. 086, 2.171, 2.224, 2.227, 2.425, 2.595, 3.220.

Data Set 2 : 0.101, 0.332, 0.403, 0.428, 0.457, 0.550, 0.561, 0.596, 0.597, 0.645, 0.954, 0.674, 0.718, 0.722, 0.725, 0.732, 0.775, 0.814, 0.816, 0.818, 0.824, 0.859, 0.875, 0.938, 0.940, 1.056, 1.117, 1.128, 1.137, 1.137, 1.177, 1.196, 1.230, 1.325, 1.339, 1.345, 1.420, 1.423, 1.435, 1.443, 1.464, 1.472, 1.494, 1.532, 1.546, 1.577, 1.608, 1.635, 1.693, 1.701, 1.737, 1.754, 1.762, 1.828, 2.052, 2.071, 2.086, 2.171, 2.224, 2.227, 2.425, 2.595, 3.220.

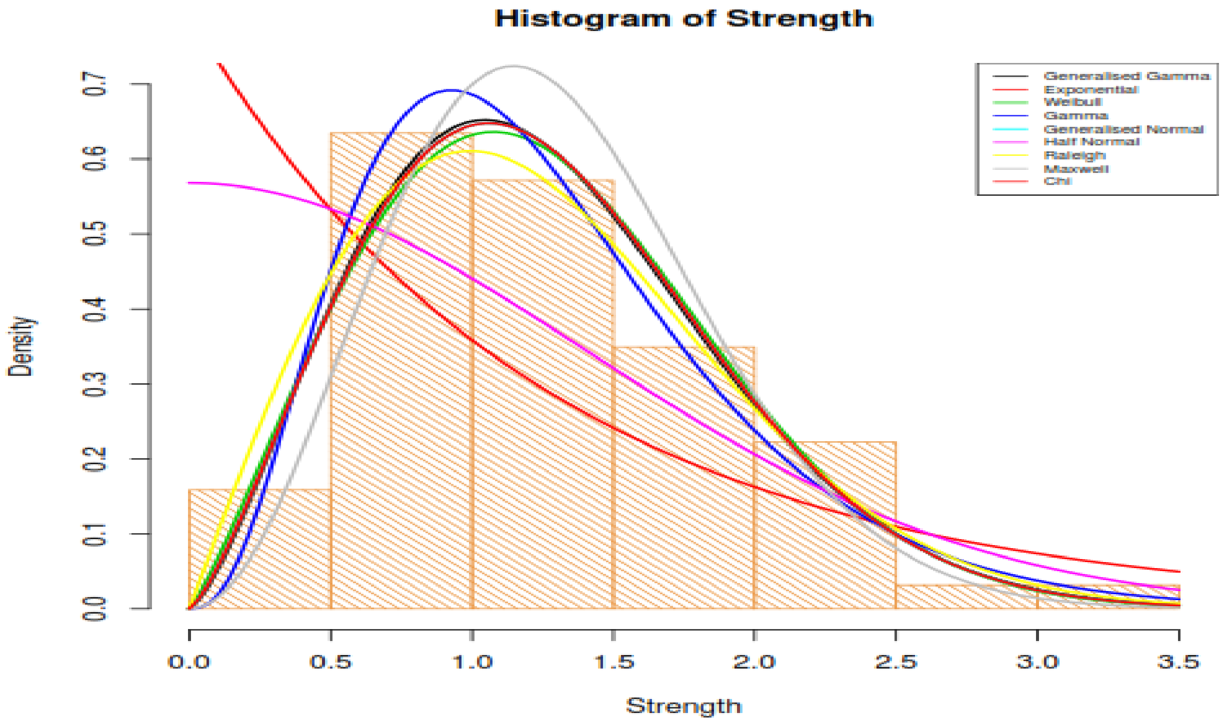
On the base of this data, we have made comparison between different cases of the Generalized Gamma distribution with its special cases, in terms of $-2\log(l)$, AIC, BIC and AICC.

The AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected) and BIC (Bayesian information criterion) of a model having p parameters are given by using the following expressions.

$$AIC=2p-2\log(l)\ \ AICC=AIC+\frac{2p(p+1)}{n-p-1}\ \text{ and }\ BIC=p\log(n)-2\log(l)$$

Table 5.1: MLE estimates, AIC, BIC and AICC values of real life data set 1.

Distribution	MLE Estimates with Standard error			-logl	$\hat{H}$	AIC	AICC	BIC
	d	b	k					
GG distribution	1.8968 (0.7878)	1.2306 (0.6041)	1.2613 (0.8689)	55.75	0.8849	117.5057	117.7091	123.9351
Exponential distribution	1	1.2640 (0.1592)	1	77.77	1.2346	157.5258	157.5586	159.6689
Weibull distribution	2.1755 (0.2112)	1.4276 (0. 0871)	1	55.81	0.8859	115.6288	115.7288	119.9151
Gamma distribution	1	0.3379 (0. 0617)	3.7399 (0.6387)	56.60	0.8985	117.207	117.307	121.4932
GN distribution	2	1.3049 (0.1287)	1.1585 (0.1840)	55.76	0.8851	115.52	115.6224	119.80
Half-Normal distribution s= $\sqrt{\frac{b^2}{2}}$	2	s=1.4046 (0.1251)	0.5	67.12	1. 0656	136.25	136.2919	138.40
Rayleigh distribution s= $\sqrt{\frac{b^2}{2}}$	2	s=0.9932 (0. 0625)	1	56.17	0.8917	**114.34**	114.3794	116.48
Maxwell distribution	2	1.1468 (0. 0589)	3/2	57.18	0.9077	116.36	116.4015	118.51
Chi	2	1.3049 (0.1287)	2.3170 (0.3681)	56.76	0.9009	115.52	115.5552	119.80

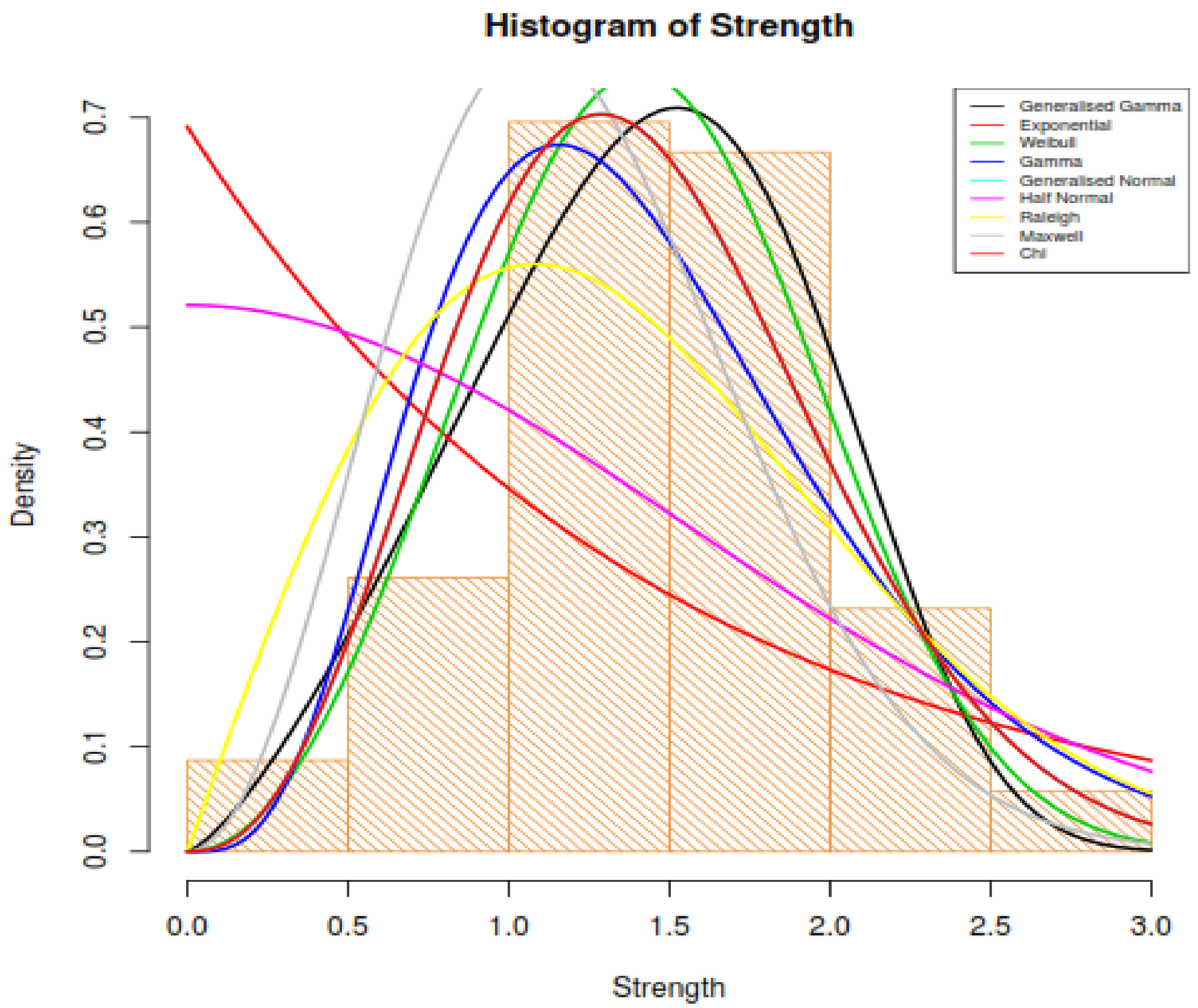


From the above table and graph, it has been observed that the Rayleigh Distribution have the smallest AIC and BIC values as compared to other family of GG Distribution. Hence we concluded that the Rayleigh Distribution gives better result and estimates as compared to GG Distribution in real life data set I.

Table 5.2: MLE estimates, AIC, BIC and AICC values of real life data II.

Distribution	**MLE Estimates with Standard error**			**-logl**	$\hat{H}$	**AIC**	**CAIC**	**BIC**
	d	b	K					
GG distribution	4.8231 (1.4208)	1.9824 (0.1907)	0.4889 (0.2071)	52.7275	0.7642	111.2705	111.4551	117.9728
Exponential distribution	1	1.4469 (0.1741)	1	94.4937	1.3694	190.9875	191. 0173	193.2216
Weibull distribution	3. 0393 (0.2922)	1.6036 (0. 066)	1	53.9576	0.7819	111.9152	112. 0061	116.3834
Gamma distribution	1	0.2907 (0. 0504)	4.9765 (0.8202)	63.1758	0.9156	130.3517	130.4426	134.8199
GN distribution	2	1.1735 (0.1065)	1.7033 (0.2664)	56.6499	0.8210	117.2998	117.3907	121.768
Half-Normal distribution s= $\sqrt{\frac{b^2}{2}}$	2	1.531 (0.1303)	0.5	79.4983	1.1521	160.9967	161. 0265	163.2308
Rayleigh distribution s= $\sqrt{\frac{b^2}{2}}$	2	1. 083 (0. 0651)	1	61.6987	0.8942	125.3597	125.4273	127.5938

Maxwell distribution	2	1. 0830 (0. 0651)	3/2	56.9685	0.8256	115.937	115.9668	118.1711
Chi	2	1.1735 (0.1065)	3.4066 (0.5328)	57.6499	0.8355	117.2998	117.3296	121.768



From the above table, it has been observed that the Generalized Gamma Distribution have the smallest AIC and BIC values as compared to other family of GG Distribution. Hence we concluded that the Generalized Gamma distribution gives better result and estimates in real life data set II.

5.2 Simulation Study of Generalized Gamma Distribution:

For description of this manner, we generate different random samples of size 25, 50 and 100 from the Generalized Gamma distribution, a simulation study is carried out 10,000 times for each values of (λ, β, k). In our simulation study, we choose a sample size of n=25, 50 and 100 to represent small, medium and large data set.

Table 5.3: MLE estimates, AIC, BIC and AICC values of simulated data (25, 1, 1, 1.5).

Distribution	MLE Estimates with Standard error			-logl	$\hat{H}$	AIC	BIC	CAIC
	d	b	k					
GG distribution	0.5678 (0.2420)	0.0433 (0.1282)	7.4685 (6.1341)	33.0295	1.321	70.059	73.716	71.16
Exponential distribution	1	1.6280 (0.3256)	1	37.38	1.4952	76.36	77.58	76.44

Weibull distribution	1.6741 (0.2560)	1.8343 (0.2321)	1	32.75	1.31	69.51	71.94	69.77
Gamma distribution	1	0.6285 (0.1849)	2.590 (0.6908)	32.22	1.28	**68.44**	70.88	68.71
GN distribution	2	2.1690 (0.3596)	0.7915 (0.1929)	33.0	1.32	70.00	72.44	70.27
Half-Normal distribution $s=\sqrt{\frac{b^2}{2}}$	2	1.9298 (0.2927)	0.5	34.812	1.39	71.624	72.381	71.71
Rayleigh distribution $s=\sqrt{\frac{b^2}{2}}$	2	1.3646 (0.1364)	1	33.49	1.33	68.98	70.20	69.06
Maxwell distribution	2	1.575 (0.1286)	3/2	37.15	1.48	76.30	77.52	76.38
Chi	2	2.1690 (0.3596)	1.5831 (0.3859)	33.0	1.32	70.00	72.44	70.27

From the above table, it has been observed that the Gamma Distribution have the smallest AIC and BIC values as compared to other family of GG Distribution. Hence we concluded that the Gamma Distribution gives better result and estimates as compared to GG Distribution in simulated data generated from Generalized Gamma Distribution with (25,1,1,1.5).

Table 5.4: MLE estimates, AIC, BIC and AICC values of Simulated Data (50, 1, 1, 1.5)

Distribution	**MLE Estimates with Standard error**			**-logl**	$\hat{H}$	**AIC**	**BIC**	**CAIC**
	d	b	k					
GG distribution	1.7632 (1.2044)	2.4689 (1.5737)	0.6867 (0.6811)	72.39	1.447	150.78	156.52	151.040
Exponential distribution	1	1.6890 (0.2388)	1	76.205	1.524	154.41	156.32	154.451
Weibull distribution	1.3950 (1.021)	1.8528 (1.5623)	1	72.455	1.4491	148.91	152.74	149.037
Gamma distribution	1	1.004 (0.2145)	1.6812 (0.3086)	72.72	1.4544	149.44	153.26	149.567
GN distribution	2	2.7189 (0.3384)	0.5829 (0.0976)	72.405	1.448	148.81	152.63	148.937
Half-Normal distribution s	2	2.0759 (0.2075)	0.5	72.81	1.456	**147.62**	149.53	147.661

Rayleigh distributions	2	1.4679 (0.1037)	1	78.47	1.5694	158.94	160.86	158.98
Maxwell distribution	2	1.6949 (0.097)	3/2	93.635	1.8727	189.27	191.18	189.31
Chi	2	2.7189 (0.3394)	1.1658 (0.1952)	72.405	1.4481	148.81	152.63	148.93

From the above table, it has been observed that the Half Normal Distribution have the smallest AIC and BIC values as compared to other family of GG Distribution. Hence we concluded that the Half Normal Distribution gives better result and estimates as compared to GG Distribution in simulated data generated from Generalized Gamma Distribution with (50,1,1,1.5).

Table 5.5: MLE estimates, AIC, BIC and AICC values of simulated data (100, 1, 1, 1.5).

Distribution	**MLE Estimates with Standard error**			**-logl**	$\hat{H}$	**AIC**	**BIC**	**CAIC**
	d	b	k					
GG distribution	1.1579 (0.4738)	1.1686 (0.9735)	1.4503 (0.9962)	134.45	1.344	274.90	282.72	274.94
Exponential distribution	1	1.5523 (0.1552)	1	143.975	1.439	289.95	292.56	289.97
Weibull distribution	1.7149 (0.1252)	1.4449 (0.1125)	1	134.62	1.346	273.24	278.45	273.30
Gamma distribution	1	0.8375 (0.1255)	1.8534 (0.2422)	134.51	1.345	**273.03**	278.24	273.09
GN distribution	2	2.4219 (0.2110)	0.6181 (0.0736)	135.48	1.354	274.96	280.17	280.23
Half-Normal distributionS	2	1.9041 (0.1346)	0.5	136.98	1.369	275.96	278.56	275.98
Rayleigh distributionS	2	1.3464 (0.0673)	1	144.845	1.448	291.69	294.30	291.71
Maxwell distribution	2	1.5547 (0.6347)	3/2	171.71	1.717	345.42	348.033	345.45
Chi	2	2.4219 (0.2110)	1.2362 (0.1472)	135.48	1.354	274.96	280.17	280.23

From the above table, it has been observed that the Gamma Distribution have the smallest AIC and BIC values as compared to other family of GG Distribution. Hence we concluded that the Gamma Distribution gives better result and estimates as compared to GG Distribution in simulated data generated from Generalized Gamma Distribution with (100, 1, 1, and 1.5).

Conclusion: In this research work, we have proposed a probabilistic model called as Generalized Gamma Distribution. Various structural properties and information measures of GGD are defined. The estimation of parameters of said model are obtained by using the Maximum Likelihood Estimator. Finally the potentiality of the GGD family has been shown by fitting it to the real life data sets and Simulated Data sets. It is quite clear from the statistical analysis that Generalized Gamma, Gamma Distribution, Half Normal Distribution and Rayleigh Distribution offers a better fit, we therefore strongly advice to the practitioners to use our proposed model.

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