THE TRANSIENT BEHAVIOUR OF A MULTI-SERVER POISSON QUEUE – A NUMERICAL APPROACH

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ABSTRACT: In this paper ,We introduce An alternative method which enables to compute the transient solution avoiding eigenvalues computation and also requires less CPU time. Moreover, the present paper is devoted to show that the proposed approach to the busy period analysis of a multichannel Markovian queue and applied to $M|M|1|N$ queue and $M|M|c$ queue can be generalized to compute the transient state probabilities and other parameters for a finite and infinite multi server Markovian queue. The effect of traffic intensity on the model's parameters such as, average queue in the system, idle probability and overflow probability is presented. The effect of initial condition on the transient state and these parameters is also given

Keywords:- Markovian queueing system, Bessel functions, transient probabilities, Busy Period Distribution .

1.1 Introduction

Several methods have been proposed for the study of the transient behaviour of the Markovian queueing system. In most cases the derived expressions are complicated because of the fact that these often use Bessel functions and Laplace transforms, see Saaty (1961), Cohen (1982) and Gross and Harris (1974). The use of Bessel functions leads to complex numerical solutions. Ever since Erlang (1909) developed the basic foundations of the theory of queues in connection with telephony, queues have been under intensive investigation by workers in a variety of fields. Later and after almost half a century, Clarke (1953) and Morse (1955) succeeded with the simple Markovian queue with infinite waiting room capacity, they obtained the transient state probabilities of $M|M|1$ in terms of Bessel functions. Lederman and Reuter (1954) studied the $M|M|1$ queue using spectral methods. Champernowne (1956) used the generating functions technique coupled with Laplace transforms. Saaty (1960) presents the $M|M|c$ queue and has obtained the Laplace transform of the distribution of the queue length, but inverted it specifically only for the case c = 2. Heathcote and Winer (1969) treated certain queues in transient state from which $M|M|1$ queue can be found. Rothkopf Oren (1979) and Clark (1981) have studied the continuous-time transient case for the generalized $M|M|c$ queues with time-dependent arrival or service rates; although these results are approximations, they appear to be accurate in most cases and do allow for non-stationary in the model parameters, which would provide added modeling flexibility. Van Doom (1981) treated the spectral function of continuous-time transient case for the $M|M|c$ queue. Whitt (1981) and Halfin and Whitt (1981) considered the $M|M|c$ queue. Further work relevant to the $M|M|c$ queue includes Pegden and Rosenshine (1982), Morisaku (1976), many papers of Abate and Whitt (1987), a& well as the papers by Parthasarathy (1987) and Bacceli & Massey (1989). Kelton and Law (1985) considered the $M|M|c$ queue with an arbitrary number of customers present in the system, at time zero. They obtained probabilities in a relatively simple closed form that can be used to evaluate exactly several measures of system performance including the expected delay in queue of each arriving customer a numerical examination is carried out to see how the choice of initial condition affects the nature of convergence of the expected delays on their steady-state values In 1989 Parthasarathy and Sharaf Ali derived an elegant time-dependent solution for the number in the $M|M|c$ queueing system in a direct way. Further, they obtained explicit expressions for the system size when c = 1,2,3 and 4. Recently, Leguesdron et.al. (1993) deal with the transient probabilities of the $M|M|1$ queue. Sharma (1990) deployed a probability argument, hitherto though intractable, to obtain a new form without Bessel function for the transien to rob-abilities of the initially empty $M|M|1$ queueing system. Conolly and Langaris (1993) developed an alternative series which isolates the steady-state component for all values of traffic intensity and which turns out to be computationally superior. In 1997, Sharma and Bunday introduced a new approach to discuss the transient state of a single server queue. However, a review of queueing literature reveals that the transient behaviour of this model with infinite waiting space has been studied by many workers using different techniques but a very little attention has been paid to obtaining transient solution for the model with finite waiting space i.e. $M|M|c|N$ queue. The transient behaviour of $M|M|1|N$ queue for general N has been discussed by Takacs (1962) and Morse (1958).

Sharma and Gupta (1982) and Sharma and Shobha (1988) have found the closed form solution of $M|M|1|N$ and $M|M|2|N$ queues respectively. Sharma and Dass (1988) studied the transient behaviour of an $M|M|c|N$ queue. The state probabilities at finite time have been found in the closed form. Later (1990) Sharma summarized these results and others related to Markovian queues in a book. However, the solution depends on the eigenvalues of a symmetric to-diagonal matrix which can't be computed easily for large values of the waiting room capacity N as these become close to each other and may be identical. Also, this method requires a lot of CPU time. In this paper ,We introduce An alternative method which enables to compute the transient solution avoiding eigenvalues computation and also requires less CPU time. Moreover, the present paper is devoted to show that the proposed approach to the busy period analysis of a multichannel Markovian queue and applied to $M|M|1|N$ queue and $M|M|c$ queue can be generalized to compute the transient state probabilities and other parameters for a finite and infinite multi server Markovian queue. The effect of traffic intensity on the model's parameters such as, average queue in the system, idle probability and overflow probability is presented. The effect of initial condition on the transient state and these parameters is also given.

1.2 Model and Solution

We consider here a system consisting of c counters. The units arrive at the system in a Poisson stream-with mean arrival rate A and the service time distribution at each c is negative exponential with the same parameter j. The waiting room capacity is taken limited to N—c places, i.e. the maximum number of customers in the system is restricted to N. We also assume that the system starts with 'i' customers and has FIFO discipline. Scaling the time so that the mean service time 1/cs is the time unit we get $\tau = c\mu t$ and $\rho = \lambda/c\mu$. Let the transient probability $p_n(t) = pr\{Q(t) = n|\, Q(0) = i\}$ be the probability that there are n units present in the system at any time t given the system starts with 'i' units at time t = 0. Our differential difference equations become

$$P'_o(\tau) = -\rho P_o(\tau) + \tfrac{1}{c}P_1(\tau)$$

$$P'_n(\tau) = \rho P_{n-1}(\tau) - \left(\rho + \tfrac{n}{c}\right)P_n(\tau) + \left(\tfrac{n+1}{c}\right)p_{n+1}(\tau),\ 1 \le n \le c-1$$

$$P'_n(\tau) = \rho P_{n-1}(\tau) - (1+\rho)P_n(\tau) + P_{n+1}(\tau) \qquad , c \le n \le N-1$$

$$P'_N(\tau) = \rho P_{N-1}(\tau) - P_N(\tau)$$

(1.2.1)

where

$$P'_n(\tau) = \tfrac{d}{d\tau}P_n(\tau) \text{ and } P_n(0) = \delta_{i,n}$$

The stationary solution is found by using the fact that $P_n(\tau)$ is independent of time i.e. $\frac{dP_n(\tau)}{d\tau} = 0$ (see Gross and Harris (1974) and Sharma (1990)). The stationary solution can be written as

Case I : For P = λ/cμ ≠ 1, we have

1. If $0 \le i \le c-1$

$$P_n = \begin{cases} \frac{(\rho c)^n}{n!}P_0 & ,0 \le n \le i \\ \rho^n\left[\frac{c^n}{n!}(1 - \sum_{k=i+1}^{n}\delta_{ck}) + \frac{c^n}{c!}\sum_{k=i+1}^{n}\delta_{ck}\right]P_0 & ,i+1 \le n \le N \end{cases}$$

2. If $c \le i \le N$

$$P_n = \begin{cases} \frac{(\rho c)^n}{n!}P_0 & ,0 \le n \le c-1 \\ \frac{\rho^n c^n}{c!}P_0 & ,c \le n \le N \end{cases}$$

where

$$P_0 = \left[\sum_{k=0}^{c-1}\frac{(\rho c)^k}{k!} + \frac{(\rho c)^c}{c!}(1-\rho^{N+1-c})(1-\rho)^{-1}\right]^{-1} \qquad (1.2.2)$$

Case II : For ρ = λ/cμ = 1, we get

1. If $0 \le i \le c-1$

$$P_n = \begin{cases} \frac{c^n}{n!}P'_0 & ,0 \le n \le i \\ \left[\frac{c^n}{n!}(1 - \sum_{k=i+1}^{n}\delta_{ck}) + \frac{c^c}{c!}\sum_{k=i+1}^{n}\delta_{ck}\right]P'_0 & ,i+1 \le n \le N \end{cases}$$

2. If $c \le i \le N$

$$P_n = \begin{cases} \frac{c^n}{n!}P'_0 & ,0 \le n \le c-1 \\ \frac{c^c}{c!}P'_0 & ,c \le n \le N \end{cases}$$

where

$$P'_0 = \left[\sum_{k=0}^{c-1}\frac{c^k}{k!} + \frac{c^c}{c!}(N-c+1)\right]^{-1} \qquad (1.2.3)$$

We assume that transient probability $p_n(t)$ is given by

$$P_n(\tau) = e^{-(1+\rho)\tau}\,\rho^{n-i}\sum_{m=0}^{\infty}a(m,n)\frac{\tau^m}{m!}\ P = \lambda c\mu\, i, n = 0,1,2,\ldots,N \quad (1.2.4)$$

Differentiating both sides w.r.t. , we get

$P_n(\tau) = e^{-(1+\rho)\tau} \rho^{n-i} \sum_{m=0}^{\infty}[a(m+1,n) - (1+\rho)a(m,n)] \frac{\tau^m}{m!}$ (1.2.5)

Substituting (1.2.4) - (1.2.5) into (1.2.1), we obtain

$$a(m+1,0) = a(m,0) + \frac{\rho}{c} a(m,1)$$

$$a(m+1,n) = a(m,n-1) + \left(1 - \frac{n}{c}\right) a(m,n) + \left(\frac{n+1}{c}\right)\rho . a(m,n+1), 1 \le n \le c-1$$

$$a(m+1,n) = a(m,n-1) + \rho a(m,n+1) \ , c \le n \le N-1$$

$$a(m+1,N) = a(m,N - 1) + \rho a(m,N)$$

with

$$a(0,n) = \delta_{in}, \quad (1.2.6)$$

where δ_{in}, is the usual Kronecker delta.

First we establish by induction that for any non-negative integers m and N, we have

$$\sum_{n=0}^{N} a(m,n)\rho^{n-i} = (1 + \rho)^m, \rho = \frac{\lambda}{c\mu} \quad (1.2.7)$$

For simple value of N = c = 1, m = 0, 1, 2, ···, we get

$$a(m+1,0) = a(m,0) + \rho a(m,1)$$

$$a(m+1,1) = a(m,0) + \rho a(m,1)$$

$$a(0,n) = \delta_{in} \quad , i,n = 0,1.$$

If i = 0 implies a(0,0) = 1, a(0,1) = 0

$$a(1,0) = a(1,1) = 1$$

$$a(2,0) = a(2,1) = 1 + \rho$$

so that

$$a(m,n)\rho^n = (1+\rho)^{m-1} + \rho(1+\rho)^{m-1}$$

$$= (1+\rho)(1+\rho)^{m-1} = (1+\rho)^m$$

If i = 1 implies a(0,0) = 1, a(0,1) = 0

$$a(1,0) = a(1,1) = 1$$

$$a(2,0) = a(2,1) = \rho(1+\rho)$$

$$a(m,0) = a(m,1) = \rho(1+\rho)^{m-1} \ m = 1,2,\cdots$$

$$\sum_{n=0}^{1} a(m,n)\rho^{n-i} = a(m,0)p^{-1} + a(m,1)$$

$= (1+\rho)^{m-1} + \rho(1+\rho)^{m-1} = (1+\rho)^m$, m = 0,1,2…

For N = 2, m = 0,1,2,…, we obtain :

(a) c = 1

$$a(m+1,0) = a(m,0) + \rho a(m,1)$$

$$a(m+1,1) = a(m,0) + \rho a(m,2)$$

$$a(m+1,2) = a(m,1) + \rho a(m,2)$$

$$a(0,n) = \delta_{in.}$$

$If\ i = 1 \Rightarrow a(0,0) = 1, a(0,n) = 0\ \forall\ n \neq 0$

$$\sum_{n=0}^{2} a(1,n)\rho^{n} = a(0,0) + \rho a(0,0) = 1 + \rho$$

$$\sum_{n=0}^{2} a(2,n)\rho^{n} = a(2,0) + a(2,1)\rho + a(2,2)p^{2}$$

$$= 1 + \rho + \rho + \rho^{2} = (1+\rho)^{2}.$$

$If\ i = 1 \Rightarrow a(0,1) = 1, a(0,n) = 0\ \forall\ n \neq 1$

$$\sum_{n=0}^{2} a(2,n)\rho^{n-1} = a(1,0)\rho^{-1} + a(1,1)\rho + a(2,2)\rho$$

$$= a(0,1) + \rho a(0,1) = 1 + \rho,$$

and

$$\sum_{n=0}^{2} a(2,n)\rho^{n-1} = a(2,0)p^{-1} + a(2,1) + a(1,2)\rho$$

$$= \big(a(1,0) + \rho a(1,1)\big)\rho^{-1} + \big(a(0,1) + \rho a(1,2)\big)$$

$$+ \big(a(1,1) + \rho a(1,2)\big)\rho$$

$$= \rho \cdot \rho^{-1} + \rho + \rho + \rho^{2} = (1+\rho)^{2}$$

$If\ i = 2 \Rightarrow a(0,2) = 1, a(0,n) = 0\ \forall\ n \neq 2$

$$\sum_{n=0}^{2} a(1,n)\rho^{n-2} = a(1,0)\rho^{-2} + a(1,1)\rho^{-1} + a(1,2)$$

$$= \big(a(0,0) + \rho a(0,1)\big)\rho^{-2} +$$

$$\big(a(0,0) + \rho a(0,2)\big)\rho^{-1} +$$

$$\big(a(0,1) + \rho a(0,2)\big)$$

$$= 1 + \rho$$

and

$$\sum_{n=0}^{2} a(2,n)\rho^{n-2} = a(2,0)\rho^{-2} + a(2,1)\rho^{-1} + a(2,2)$$

$$= \big(a(1,0) + \rho a(1,1)\big)\rho^{-2} + \big(a(1,0) + \rho a(1,2)\big)^{-1}$$

$$+\big(a(1,1) + \rho a(1,2)\big)$$

$$= (1+\rho)^{2}$$

(a) c = 2

$$a(m+1,0) = a(m,0) + \frac{\rho}{2} a(m,1)$$

$$a(m+1,1) = a(m,0) + \frac{1}{2} a(m,1) + \rho a(m,2)$$

$$a(m+1,2) = a(m,1) + \rho a(m,2)$$

$$a(0,n) = \delta_{in.}$$

$If\ i = 0 \Rightarrow a(0,0) = 1, a(0,n) = 0\ \forall\ n \neq 0$

$$\sum_{n=0}^{2} a(1,n)\rho^{n} = a(1,0) + \rho a(1,1) = 1 + \rho$$

and

$$\sum_{n=0}^{2} a(2,n)\rho^n = a(2,0) + a(2,1)\rho + a(2,2)p^2$$

$$= a(0,1) + \frac{\rho}{2}a(1,1) + \left(a(1,0) + \frac{1}{2}a(1,1) + \rho a(1,2)\right)\rho + \left(a(1,1) + \rho a(1,2)\right)\rho^2$$

$$= a(0,0) + \frac{\rho}{2}a(0,0) + \left(a(0,0) + \frac{1}{2}a(0,0)\rho + a(0,0)\right)\rho^2$$

$$= 1 + \tfrac{\rho}{2} + \rho + \tfrac{\rho}{2} + \rho^2 = (1+\rho)^2$$

$If\ i = 1 \Rightarrow a(0,1) = 1, a(0,n) = 0\ \forall\ n \neq 1$

$$\sum_{n=0}^{2} a(1,n)\rho^{n-1} = a(1,0)\rho^{-1} + a(1,1) + a(1,2)p$$

$$= \tfrac{\rho}{2}\cdot\rho^{-1} + \tfrac{1}{2} + \rho = (1+\rho)$$

and

$$\sum_{n=0}^{2} a(2,n)\rho^{n-1} = a(2,0)\rho^{-1} + a(2,1) + a(2,2)p$$

$$= \left(a(1,0) + \tfrac{\rho}{2}a(1,1)\right)\rho^{-1} +$$

$$a(1,0) + \tfrac{1}{2}a(1,1) + \rho a(1,2) +$$

$$\left(a(1,1) + \rho a(1,2)\right)\rho$$

$$= \left(\tfrac{\rho}{2} + \tfrac{\rho}{2}\cdot\tfrac{1}{2}\right)\rho^{-1} + \tfrac{\rho}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} + \rho + \left(\tfrac{1}{2} + \rho\right)\rho$$

$$= (1+\rho)^2$$

$If\ i = 2 \Rightarrow a(0,2) = 1, a(0,n) = 0\ \forall\ n \neq 2$

$$\sum_{n=0}^{2} a(1,n)\rho^{n-2} = a(1,0)\rho^{-2} + a(1,1)\rho^{-1} + a(1,2)$$

$$= \rho\cdot\rho^{-1} + \rho = (1+\rho)$$

and

$$\sum_{n=0}^{2} a(2,n)\rho^{n-2} = a(2,0)\rho^{-2} + a(2,1)\rho^{-1} + a(2,2)$$

$$= \tfrac{\rho^2}{2}\cdot\rho^{-2} + \left(\tfrac{\rho}{2}\cdot\rho^2\right)\rho^{-1} + \rho + \rho^2$$

$$= (1+\rho)^2$$

Which establish the result (1.2.7) for the simple value N = 1, 2; m = 0,1,2.

Now, assume that N > 2 and suppose the result (1.2.7) is true for some integer m. By using mathematical induction we shall show (1.2.7) is true for $m + 1$. Multiplying the set of equations (1.2.6) by $\rho^{-i}, \rho^{n-i}, \rho^{N-i}$ respectively and taking the summation over $n, 0 \leq n \leq N$, we get

$$\sum_{n=0}^{N} a(m+1,n)\rho^{n-i} = \left(a(m,0) + \frac{\rho}{c}a(m,1)\right)\rho^{-i}$$

$$+ \sum_{n=1}^{c-1}\left(1 - \frac{n}{c}\right)a(m,n)\rho^{n-i} + \sum_{n=1}^{c-1} a(m,n-1)\rho^{n-i}$$

$$+ \sum_{n=1}^{c-1}\left(\frac{n+1}{c}\right)a(m,n+1)\rho^{n+1-i} + \sum_{n=1}^{N-1} a(m,n-1)\rho^{n-i}$$

$$+\rho\sum_{n=1}^{N-1} a(m,n+1)\rho^{n-i} + a(m,N-1)\rho^{N-i} + a(m,N)\rho^{N+1-i}$$

$$= \left(a(m,0) + \frac{\rho}{c}a(m,1)\right)\rho^{-i} - \sum_{n=1}^{c-1}\left(\frac{n}{c}\right)a(m,n)\rho^{n-i}$$

$$+\sum_{n=1}^{c-1} a(m,n)\rho^{n-i} + \sum_{n=1}^{c-2} a(m,n)\rho^{n+1-i}$$

$$+\sum_{n=2}^{c}\left(\frac{n}{c}\right)a(m,n)\rho^{n-i} + \sum_{n=c-1}^{N-2} a(m,n)\rho^{n+1-i}$$

$$+\sum_{n=c+1}^{N-2} a(m,n)\rho^{n-i} + a(m,N-1)\rho^{N-i} + a(m,N)\rho^{N+1-i}$$

$$= \left(a(m,0) + \frac{\rho}{c}a(m,1) - \frac{\rho}{c}a(m,1)\right)\rho^{-i} - \sum_{n=2}^{c-1}\left(\frac{n}{c}\right)a(m,n)\rho^{n+1-i}$$

$$+\sum_{n=1}^{c-1} a(m,n)\rho^{n-i} + \sum_{n=0}^{c-2} a(m,n)\rho^{n+1-i}$$

$$\sum_{n=2}^{c-1}\left(\frac{n}{c}\right)a(m,n)\rho^{n-i} + a(m,c)\rho^{c-i}$$

$$+\sum_{n=c-1}^{N-2} a(m,n)\rho^{n+1-i} + \sum_{n=c+1}^{N} a(m,n)\rho^{n-i}$$

$$+ a(m,N-1)\rho^{N-i} + a(m,N)\rho^{N+1-i}$$

$$= \sum_{n=0}^{N} a(m,n)\rho^{n-i} + \sum_{n=0}^{N} a(m,n)\rho^{n+1-i}$$

$$= (1+\rho)\sum_{n=0}^{N} a(m,n)\rho^{n-i} = (1+\rho)(1+\rho)^m$$

$$= (1+\rho)^{m+1}$$

which establish the result.

According to this result, we obtain

$$\sum_{n=0}^{N} P_n(t) = e^{-(1+\rho)\tau}\sum_{m=0}^{\infty}\left(\sum_{m=0}^{\infty} a(m,n)\,\rho^{n-i}\right)\frac{\tau^m}{m!}$$

$$= e^{-(1+\rho)\tau}\sum_{m=0}^{\infty}(1+\rho)^m\frac{\tau^m}{m!}$$

$$= e^{-(1+\rho)\tau}\cdot e^{(1+\rho)\tau} = 1.$$

1.3 Busy Period Distribution for $M|M|c|N$

The busy period during which c channels are busy in an $M|M|c|N$ queueing system is defined as the time from the instant of arrival of c units that make c channels busy to the first subsequent instant when one of the c channels becomes empty.

We define

$$q_n(t) = pr\,\{Q(t) = n|Q(0) = i, c \le i \le N\} \qquad (1.3.1)$$

and let $q_n(0) = \delta_{in}, c \le i \le N$ where δ_{in} is Kronecker delta, and (c - 1)th state is an absorbing state, then

$\frac{d}{dt}q_{c-1}(t)$ = the density function of the busy period b(t) (1.3.2) and

$q_{c-1}(t)$ = the c.d.f. of the busy period, B(t) (1.3.3)

We have the following transition equations for the system,

$$q'_{c-1}(t) = c\mu q_c(t)$$

$$q'_c(t) = -(\lambda + c\mu)q_n(t) + c\mu q_{n+1}(t)$$

$$q'_n(t) = -(\lambda + c\mu)q_n(t) + \lambda q_{n+1}(t) + c\mu q_{n+1}(t), c+1 \le n \le N-1$$

$$q'_N(t) = -c\mu P_N(t) + \lambda P_{N-1}(t) \qquad (1.3.4)$$

with

$$q_n(0) = \delta_{in} \qquad \text{and} \qquad q'_n(t) = \frac{d}{dt} q_n(t).$$

Let

$$q_n(\tau) = e^{-(1+\rho)\tau}\rho^{n-i}\sum_{m=0}^{\infty}(1+\rho)^m\frac{\tau^m}{m!} \qquad \rho = \frac{\lambda}{c\mu} \qquad (1.3.5)$$

which yields

$$q'_n(\tau) = e^{-(1+\rho)\tau}\rho^{n-i}\sum_{m=0}^{\infty}\left(a(m+1,n)\right) - (1+\rho)a(m,n)\frac{\tau^m}{m!} \qquad (1.3.6)$$

where .

$$q'_n(\tau) = \dot{q}_n(\tau)\cdot\frac{dt}{d\tau} = \dot{q}_n(t)\cdot 1/c\mu$$

Using (1.3.5) and (1.3.6) in (1.3.4), we easily get

$$a(m+1, c-1) = (1+\rho)a(m, c-1) + \rho a(m, c)$$

$$a(m+1, c) = \rho a(m, c+1)$$

$$a(m+1, n) = a(m, n-1) + \rho a(m, n+1), c+1 \le n \le N-1$$

$$a(m+1, N) = a(m, N-1) + \rho a(m, N)$$

and

$$\rho^{n-i}(0, n) = \delta_{in}, \quad c \le i \le N$$

$$b(t) = c\mu q_c(t)$$

$$= c\mu e^{-(\lambda+c\mu)t}\rho^{c-i}\sum_{m=0}^{\infty}a(m,c)\frac{(c\mu e)^m}{m!}, \quad \rho = \lambda/c\mu$$

and

$$B(t) = \rho^{c-i}\sum_{m=0}^{\infty}a(m,c)\frac{(c\mu)^{m+1}}{m!}\int_0^t e^{-(\lambda+c\mu)u}u^m du$$

$$= \rho^{c-i}\sum_{m=0}^{\infty}a(m,c)\frac{(c\mu)^{m+1}}{m!}I_m(t)$$

where

$$I_0(t) = \frac{1}{\lambda + c\mu}\left(1 - e^{-(\lambda+c\mu)t}\right)$$

and

$$I_m(t) = \frac{t^m}{\lambda + c\mu}e^{-(\lambda+c\mu)t} + \frac{m}{\lambda + c\mu}I_{m-1}(t), m \ge 1$$

1.4 Sharma & Dass Formula and Algorithms

As mentioned earlier our primary interest in this paper is the numerical computation of state probabilities for $M|M|c|N$ queue without eigenvalues computation or Chebychev's polynomials as showen in Sharma's book. Our curiosity was aroused by a formula by Sharma & Das (1988) which may be the only analytical form available for the transient solution of an $M|M|c|N$ queue, with c > 1 and i customers waiting at time t = 0.

Here, we recall this formula for the case c = 2 to compare it with our approach. After considerable simplification we get,

case $\rho = \lambda/2\mu \neq 1$

$$P_n(t) = \begin{cases} B + e^{-(\lambda+c\mu)t}\sum_{j=1}^{N}\left[\frac{(-1)^1}{2}(1-\delta_{i0})\rho^{\frac{i}{2}}A_{ije^{-aN_j}} + \delta_{i0}A_{0je^{-aN_j\sqrt{2\mu\lambda}.t}}\right], n = 0 \\ 2B\rho^n + e^{-(\lambda+2\mu)t}\sum_{j=1}^{N}\left[(-1)^{n-i}\rho^{\frac{(n-i)}{2}}A_{ij}h_n(aN_j)e^{-aN_j\sqrt{2\mu\lambda}.t}\right], n = 1,2,\dots,i \\ 2B\rho^n + e^{-(\lambda+2\mu)t}\sum_{j=1}^{N}(-1)^{n-i}\rho^{\frac{(n-i)}{2}}A_{nj}h_i(a_{Nj})e^{-aN_j\sqrt{2\mu\lambda}.t}, \quad n = i+1, i+2, \dots, N \end{cases}$$

where

$$B = (1-\rho)[(1-\rho) + 2\rho(1-\rho^N)]^{1-}$$

$$A_{nj} = \frac{g_{N-n}(a_{Nj}) + \rho^{1/2}g_{N-n-1}(a_{Nj})}{(\rho^{1/2}+\rho^{-1/2}+a_{Nj})b_{Nj}}$$

$$A_{0j} = \left[g_N(a_{Nj}) + \left(\rho^{\frac{1}{2}} + \frac{1}{2}\rho^{-\frac{1}{2}}\right)g_{N-1}(a_{Nj}) + \frac{1}{2}g_{N-2}(a_{Nj})\right] \times \left[\left(\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}}a_{Nj}\right)b_{Nj}\right]^{-1}$$

$$b_{Nj} = \prod_{\substack{k=1 \\ k\neq j}}^{N}(a_{Nj} - a_{Nk})$$

and for $\rho = 1$, i.e. $\lambda = 2\mu$

$$P_n(t) = \frac{1}{1+2N} + e^{-2\lambda t}\sum_{j=1}^{N}\left[\frac{1}{2}(1-\delta_{i0})A'_{ije^{-aN_j\lambda t}} + \delta_{i0}A'_{0je^{-aN_j\lambda t}}\right], n = 0$$

$$P_n(t) = \frac{1}{1+2N} + e^{-2\lambda t}\sum_{j=1}^{N}\left[(-1)^{n-i}A'_{ij}h_n(a_{Nj})e^{-aN_j\lambda t}\right] \qquad, n = 1,2,\dots,i$$

$$P_n(t) = \frac{1}{1+2N} + e^{-2\lambda t}\sum_{j=1}^{N}(-1)^{n-i}A'_{nj}h_i(a_{Nj})e^{-aN_j\lambda t} \qquad, n = i+1, i+2, \dots, N$$

where

$$A'_{0j} = \left[g_N(a_{Nj}) + \frac{3}{2}g_{N-1}(a_{Nj}) + \frac{1}{2}g_{N-2}(a_{Nj})\right]\left[(2+a_{Nj})b_{Nj}\right]^{-1}$$

$$A'_{nj} = \left[g_{N-n}(a_{Nj}) + g_{N-1}(a_{Nj})\right]\left[(2+a_{Nj})b_{Nj}\right]^{-1}$$

$$g_n(x) = \begin{vmatrix} x & \sqrt{2\mu\lambda} & 0 & \cdot & \cdots & 0 \\ \sqrt{2\mu\lambda} & x & \sqrt{2\mu\lambda} & 0 & \cdots & 0 \\ 0 & \sqrt{2\mu\lambda} & x & \sqrt{2\mu\lambda} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \sqrt{2\mu\lambda} & x \end{vmatrix}_{n\times n}$$

$$h_n(x) = \begin{vmatrix} x-2\mu & \sqrt{\mu\lambda} & 0 & \cdot & \cdots & 0 \\ \sqrt{\mu\lambda} & x-\mu & \sqrt{2\mu\lambda} & 0 & \cdots & 0 \\ 0 & \sqrt{2\mu\lambda} & x & \sqrt{2\mu\lambda} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \sqrt{2\mu\lambda} & x \end{vmatrix}_{n\times n}$$

and $a_{Nj}(j = 1,2,\cdots,N)$ are real and distinct eigenvalues of the matrix

$$A = \begin{vmatrix} -\sqrt{\mu/2\lambda} & 1/\sqrt{2} & 0 & \cdot & \cdots & 0 \\ 1/\sqrt{2} & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \sqrt{2\mu\lambda} & x \end{vmatrix}_{N\times N}$$

to compute the transient probabilities and other parameters related to $M|M|c|N$ queue, the following algorithms are used:

Algorithm 1

(a) Set t_1 = time

(b) Generate the matrix A

(c) Compute the eigenvalues of A

(d) Compute $P_n(t)$ via Sharma & Dass formula

(e) Set t_2 = time

(f) Compute $CPU = t_2 - t_1$

(g) Compute the other parameters.

Algorithm 2

(a) Set t_1 = time

(h) Set $a(0,i) = 0.0$ and $a(0,j) = 0.0\ \forall\ j \neq i, j = 0,1,2,\cdots,N$

(c) Generate the coefficients $a(m,n) m = 1,2,\cdots,L,\ n = 0,1,\cdots,N$

(d) Compute $P_n(t)$ via equation (5.4)

(e) Set t_2 = time

(f) Compute CPU = $t_2 - t_1$

(g) Compute the other parameters

where L is a sufficiently large number to give the probability sum equal to 1.

1.5 Numerical Results and Discussion

To demonstrate the working of the method proposed in section 1.2 and 1.3, we have carried out extensive numerical work on computer system" for different values of t, μ, λ, c and N. The system performance measures' as, transient state probabilities, mean and variance of queue length are presented here. Moreover, the effect of traffic intensity, on these parameters is given. We have also written a computer program based on a formula of Sharma & Dass (1988) mentioned in section 1.5 and have compared it with our approach. As mentioned earlier it seems that Sharma & Dass is the only formula available in the literature with which Nye can have the comparison. In that formula they obtained the transient state distribution via the eigenvalues computation and these eigenvalues become identical for large values of room capacity which effects directly the accuracy of the transient solution. Whereas by using the proposed method in section 1.3. We are able to compute the transient solution without eigenvalues computation or Chebychev's polynomial and no restriction on the room Capacity is needed.

The numerical computation for the $M|M|c|N$ busy period c.d.f. (cumulative distribution function B(t)) is developed: Our procedure for computing the c.d.f. is given in section 1.3. The numerical integration procedure could obviously be replaced by recurrence relation which gives exact values for that integration.

- The method of calculating the transient parameters of interest and busy period distribution can be computed quite easily without deriving or applying the complete expressions for $P_n(t)$ and $b(t)$ and avoiding the computation of eigenvalues in these expressions.

- The comparison between the proposed method and Sharma & Dass formula shows that the proposed method is faster.
- The transient probabilities obtained by the proposed method are same as those in the analytical solution.
- The calculation shows that the transient probabilities decrease for large values of the room capacity whereas the mean and variance of queue length in the system increase.
- The values of system parameters such as mean and variance of queue length are increased when ρ increases.
- The values of busy period distribution decrease when ρ increases as well as when room capacity "N" and initial start "i" increase.

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