DYNAMICS OF DISCRETE TUMOR-IMMUNE MODEL

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Abstract : This paper considers a two dimensional discrete model of tumor growth. Existence of equilibrium points is established and local stability conditions are obtained. The phase portraits are obtained for different sets of parameter values. Numerical simulations are performed to illustrate the dynamics of tumor growth and they exhibit rich dynamics of the discrete model.

IndexTerms - **Cancer, Mathematical Model of Tumor Immune in Discrete System, Numerical Simulation, Stability, m Steady states.**

I. INTRODUCTION

Globally, cancer is the cause for most deaths in developed and developing countries. Roughly 75 lakhs death is caused by different types of cancer. Curing caner is a challenge faced by the medical world today. Human body contains about 50 trillion cells and they undergo division every day. Due to mutated DNA, cell growth becomes uncontrollable and after a certain period, rapid cell growth is observed. Modelling the tumor growth, capturing the response of tumors to treatment and predicting the response of tumors are the main goals of researchers. Mathematical models of growth by ODE, PDE and difference equations have played vital role in the understanding of interactions in population dynamics, ecology and infectious diseases. The interaction models are suitable to describe the tumor growth and the response of the cells to therapy[1, 2, 3, 4]. Paper [6] describes the penetration of the tumor cells by the effector cells, which causes the effector cells to become inactive.

II. DISCRETE MODEL OF IMMUNE-TUMOR SYSTEM

This paper considers the following system of difference equations to describe the interaction between two types of cells.

$$\begin{aligned} E(t+1) &= s + (1-a)E(t) + bE(t)T(t) \\ T(t+1) &= rT(t)\big(1 - cT(t)\big) + (1 - E(t))T(t) \end{aligned} \qquad (1)$$

where $E(t)$ and $T(t)$ are the densities with respect to immune effector cells and tumor cells respectively. The parameters are positive to be biologically meaningful. The first equation describes the rate of change in the effector cells population. The parameter s is the external source of the effector cells, a is the natural decay rate of the effector cells, b represents tumor deactivation rate of effectors. When the immune response is absent, tumor cells follow a logistic growth. The parameter r is the intrinsic growth rate of tumor cells and the tumor population carrying capacity is $\frac{1}{c}$.

III. EQUILIBRIA AND DYNAMICAL BEHAVIOR

The system (1) has the following three equilibrium points.

(1) $E_0 = \left(\frac{s}{a}, a\right)$ isthe tumor free equilibrium. This steady state always exist $\frac{s}{a} > 0$.

(2) $E_1 = \left(\frac{r(b-ac) - \sqrt{\Delta}}{2b}, \frac{r(b+ac)+\sqrt{\Delta}}{2rbc}\right)$

(3) $E_2 = \left(\frac{r(b-ac)+\sqrt{\Delta}}{2b}, \frac{r(b+ac)-\sqrt{\Delta}}{2rbc}\right)$

E_1 and E_2 are two endemic equilibria with $\Delta = r^2(ac-b)^2 + 4srbc > 0$.

The local stability of the system (1) is studied by computing the Jacobian matrix corresponding to each equilibrium point [5, 7]. The Jacobian matrix for the system (1) is

$$J(E,T) = \begin{bmatrix} (1-a) + bT^* & bE^* \\ -T^* & (1+r) - E^* - 2rcT^* \end{bmatrix} \qquad (2)$$

Theorem 1. *Tumor free equilibrium E_0 is locally asymptotically stable if $0 < a < 2$ and $ra < s < a(r+2)$, otherwise unstable equilibrium point.*

Proof. Jacobian matrix for the tumor free equilibrium point E_0 is given by

$$J(E_0) = \begin{bmatrix} 1-a & \frac{sb}{a} \\ 0 & (1+r) - \frac{s}{a} \end{bmatrix}$$

The eigenvalues of $J(E_0)$ are $\lambda_1 = 1 - a$and $\lambda_2 = (1+r) - \frac{s}{a}$. Thus E_0 is stable when $0 < a < 2$and $ra < s < a(r+2)$. Otherwise E_0 is unstable equilibrium point.

Theorem 2. *Endemic equilibrium* E_1*is locally asymptotically stable if* $(s+r) + \sqrt{(s-r)^2 - 4ab} < 2$*and* $(s+r) - \sqrt{(s-r)^2 - 4ab} < 2$, *otherwise unstable equilibrium point.*

*Proof.*Jacobian matrix for the endemic equilibrium point E_1 is given by

$$J(E_1) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where, $a_{11} = \frac{rc(2-a)+rb+\sqrt{\Delta}}{2rc}$, $a_{12} = \frac{r(b-ac)-\sqrt{\Delta}}{2}$, $a_{21} = \frac{r(b+ac)+\sqrt{\Delta}}{2rbc}$ and $a_{22} = \frac{2b-r(b+ac)-\sqrt{\Delta}}{2b}$. Eigenvalues of $J(E_1)$ are $\lambda_{1,2} = \frac{(s+r)\pm\sqrt{(s-r)^2-4ab}}{2}$. Thus E_1 is stable when $(s+r) + \sqrt{(s-r)^2 - 4ab} < 2$and $(s+r) - \sqrt{(s-r)^2 - 4ab} < 2$. Otherwise E_1 is unstable equilibrium point. □

Similarly, we obtain following result.

Theorem 3. *If the endemic equilibrium* E_2*exists and has non negative coordinates, then it is locally asymptotically stable.*

IV. NUMERICAL DISCUSSION

This section presents some numerical examples on the dynamics of tumor growth. Mainly, we present the time plots of the solutions E and T with phase plane diagrams for the system(1).

Example 1. *Choosing the parameters values* $s = 1.16, r = 0.32, a = 1.92, b = 0.21,$ *and* $c = 0.02$. *We obtain* $E_0 = (0.60, 0)$ *and the eigenvalues are* $\lambda_1 = -0.92$ *and* $\lambda_2 = 0.72$ *so that* $|\lambda_{1,2}| < 1$. *Hence the free equilibrium point is stable. In this case the immune effector cells population survives and the tumor cells population goes to extinction see Figure-(1).*

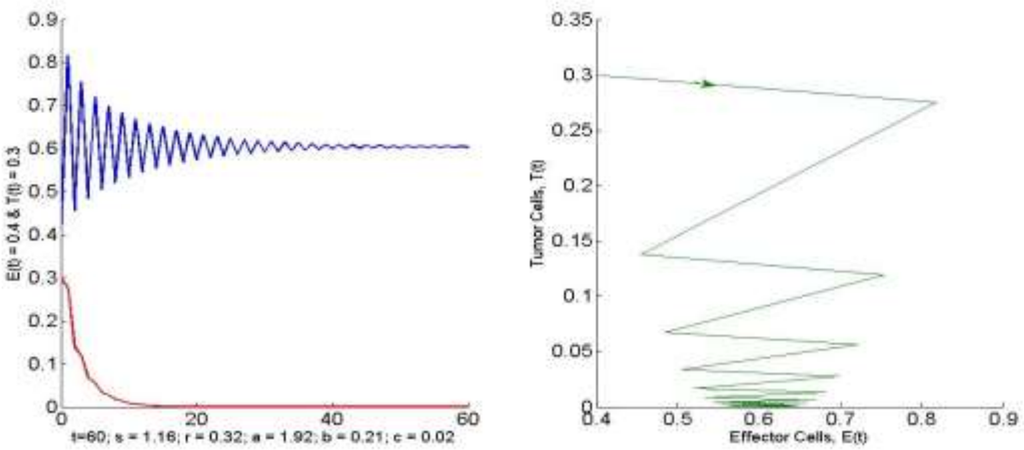

Figure 1. Time series plot and Phase Portrait are Stability at E_0

Example 2.*Figure* (2) *and Figure* (3) *display the endemic steady states for the system* (1) *with* $r = 0.60, 0.62, 0.63, 0.65$ *and parameter values* $s = 1.19, a = 1.98, b = 0.01, c = 0.12$.*Hence both endemic equilibrium point* E_1 *and* E_2 *are locally asymptotically stable.*

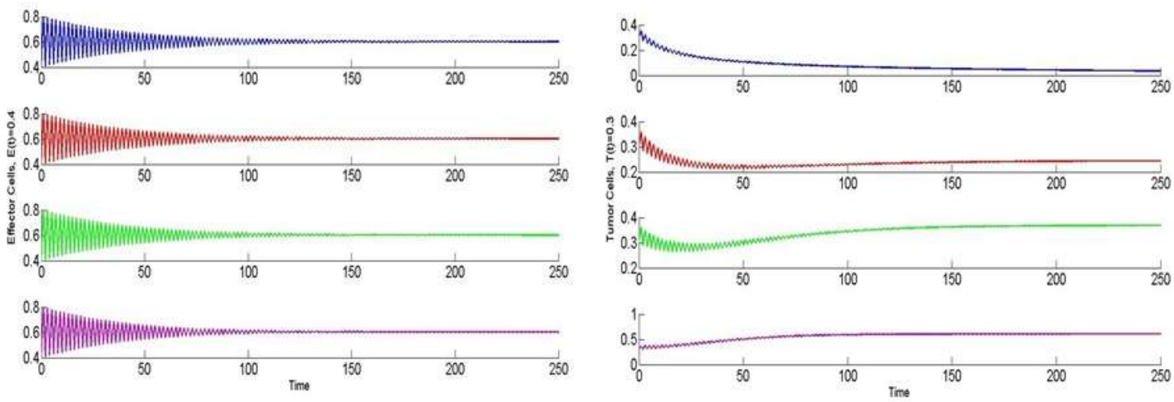

Figure 2. Time series plot for Stability at both endemic equilibrium point

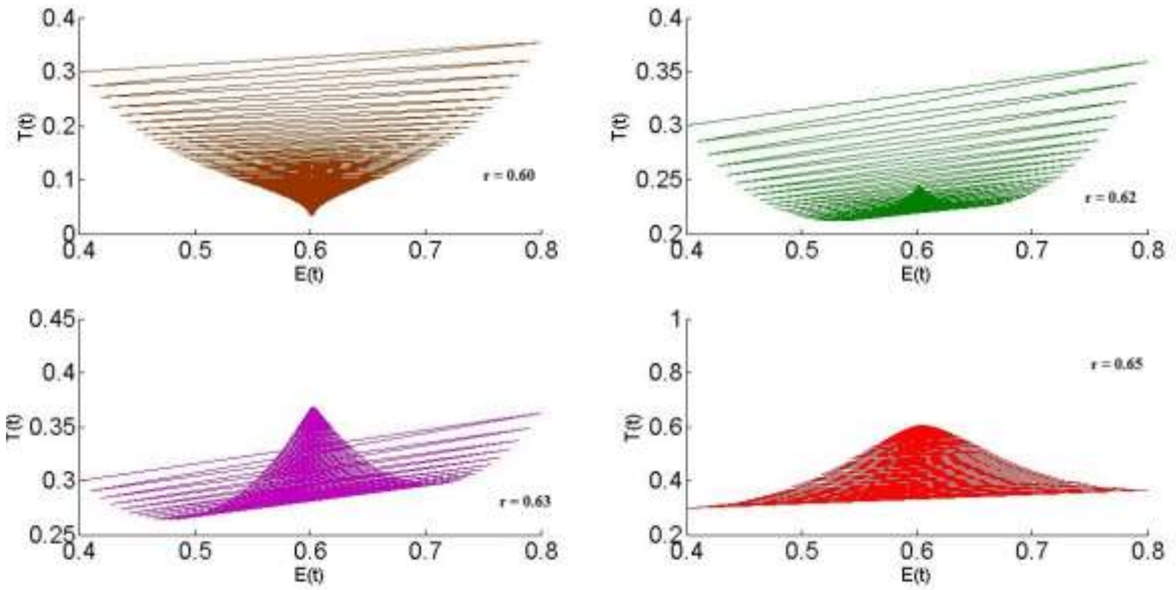

Figure 3. Phase Portrait for stability at both endemic equilibrium point

V. CONCLUSION

We have considered and investigated the stability properties of a 2-D discrete dynamical system describing tumor-immune system. Time plots and phase portraits are presented to show the extinction and presence of tumor cells.

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