Some Fixed point theorems in Complex valued metric Space using subcompatible mappings

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Abstract

The aim of the present paper is to establish fixed point theorems for subcompatible mappings.

Keywords: Complex valued metric space, Subcompatible maps, fixed point

1 Introduction

Banach contraction principle was the milestone of many research field of non-linear analysis. In 2011, the concept of complex valued metric space was introduced by Azam et.al. Banach fixed point theorem [2] in a complete metric space has been generalised in so many spaces.

Recently in 2009, using the concept of sub compatible maps, H.Bouhadjera et.al.[3] proved common fixed point theorems. In 2010 and 2011, B. Singh et.al. [13] proved fixed point theorems in Fuzzy metric space and Menger space using the concept of semi-compatibility, weak compatibility and compatibility of type (β) and weakly compatible maps. Different authors used concept of weakly compatibiliy and occasionally weakly compatiblility in different space such as fuzzy metric space, menger space etc to prove fixed point theorems.

The main purpose of this paper is to introduce concept of subcompatible maps and α–*subcompatible* maps in complex valued metric space and we obtain some fixed point theorems in complex valued metric space,.

2 Preliminary Notes

Here we will give the definitions, examples and results which will be used in the following section

Definition 1

Let be the set of complex numbers and let $z,w \in C$.

Define a partial order relation $\leq$ on C and $z,w \in C$, $z \precsim w$ if and only if Re $z \leq$ Re w and Im $z \leq$ Im w

Definition 2 *Let X be a non-empty set. Suppose that the mapping d: $X \times X$ \o ($\longrightarrow$) C is a metric on X if it satisfies the following conditions*

(CM1) $d(x,y)>0$ and $d(x,y)=0$ if and only if $x=y$

(CM2) $d(x,y)=d(y,x)$ for all $x,y \in X$

(CM3) $d(x,y) \precsim d(x,z)+d(z,y)$ for all $x,y,z \in X$ Then d is called a complex valued metric and (X,d) is a complex valued metric space.

Example 3 *Let X be a complex valued space.Consider*

$d(z,w) = i|z-w|$, Then d is a complex valued metric and (X,d) is a complex valued metric space.

Definition 4 *Let (X,d) be a complex valued metric space*

1. *If $\{x_n\}$ be a sequence in X and $x \in X$.Then $\{x_n\}$ is said to be convergent sequence, if $\{x_n\}$ coverges to $x \in X$ we denote this by $\lim_{n\to\infty} x_n = x$;*

2. *If c $\in X$ with $0 \preceq c$ there exist $n \in N$ $d(x_n,x_m) \preceq c$ where $m \in N$, then $\{x_n\}$ is said to be a Cauchy sequence;*
3. *If for every Cauchy sequence in X is convergent then (X,d) is said to be a complete complex valued metric space*

Example 5 *If (X,d) is a metric space then the metric d induces a mapping d: X×X \o ($\longrightarrow$) C, defined by $d(x,y)= |x_1 - x_2|+i|\ y_1 - y_2|$ for all $x,y \in X$. Then (X,d) is a complex valued metric space*

Definition 6 *Let f and g be self-maps on a set X, if w = fx = gx for some x $\in X$, then x is called a coincidence point of f and g, w is called a point of coincidence of f and g*

Definition 7 *Let f and g are self-maps defined on a set X.Then f and g are said to be weakly compatible if they commute at their coincidence points.*

Definition 8 *Let X be a non-empty set and f,g:X \o ($\longrightarrow$) X be mappings. A pair (f,g) is called weakly compatible if x $\in X$, fx=gx implies*

fgx= gfx

Definition 9 *Two self maps A and S of a complex valued metric space (X,d) is called weakly commuting if $d(ASx,SAx) \succeq d(Ax,Sx)$ for all $x \in X$*

Definition 10 *Two maps A and S of a complex valued metric space (X,d) is called R-weakly commuting if there exist R >0 such that $d(ASx,SAx) \succeq Rd(Ax,Sx)$*

Definition 11 *Two self maps A and S on a complex valued metric space are said to be reciprocal continous if $\lim_{n\to\infty} ASx_n = Au$ and $\lim_{n\to\infty} SAx_n = Su$*

3 Main Results

Definition 1 *Let f and g be two selfmaps of a complex valued meric space (X,d),then f and g are said to be subcompatible maps if and only if there exist a sequence x_n in X such that $d(fx_n,z)=d(gx_n,z)\preceq c$*

Theorem 3.2 *Let (X,d) be a complex valued metric space.Suppose that the mapping $f,g:X\to X$ are subcompatible and satisfies*

$d(fx,fy)\preceq \alpha d(fx,gx)+ \frac{\beta}{1+\gamma}d(fy,gy)$ and $\alpha-\gamma+\alpha\beta+\beta \leq 1$.Then f and g have a a unique common fixed point.

Proof:

Let $x_0 \in X$ be arbitrary we define a sequence such that $fx_n = gx_{n+1}$ for all n>0 Consider

$$\begin{aligned} d(gx_{n+1},gx_n) &= d(fx_n,fx_{n-1}) \\ &\leq \alpha d(fx_n,gx_n)+ \frac{\beta}{1+\gamma}d(fx_{n-1},gx_{n-1}) \\ (1-\alpha)d(gx_{n+1},gx_n) &\leq \frac{\beta}{1+\gamma}d(gx_n,gx_{n-1}) \end{aligned}$$

$$\begin{aligned} d(gx_{n+1},gx_n) &= d(fx_n,fx_{n-1}) \\ d(gx_n,gx_{n+1}) &\leq hd(gx_{n-1},gx_n) \end{aligned}$$

where $h= \frac{1}{1-\alpha}$

$$\begin{aligned} d(gx_{n+1},gx_n) &\le hd(gx_n,gx_{n-1}) \\ &\le h^2 d(gx_{n-1},gx_{n-2}) \\ &\le h^3 d(gx_{n-2},gx_{n-3}) \ldots\ldots \\ &\le h^n d(gx_0,gx_1) \end{aligned}$$

If m >n we have

$$\begin{aligned} d(y_n,y_m) &\le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+\ldots+d(y_{m-1},y_m) \\ &\le h^n d(y_0,y_1)+h^{n+1}d(y_0,y_1)+\ldots+h^{m-1}d(y_0,y_1) \\ &\le \frac{h^n}{1-h}d(y_0,y_1) \end{aligned}$$

Since $|h|<1, d(gx_n,gx_m)\to 0$ So (gx_n) is a Cauchy sequence.

Since complex valued metric space is complete, every Cauchy sequence converges.

So it converges to z.

That is $\lim_{n\to\infty} gx_n = z$.

To prove that z is a fixed point of g

For that consider as $n\to\infty, d(gz,z)\to 0$

That is z is a fixed point of g.

Now to prove that z is a fixed point of f,

For that consider

$$\lim_{n\to\infty} d(fz,z)=d(fz,gx_n)= \frac{1}{\alpha}(d(fz,fx_n)- \frac{\beta}{1+\gamma}d(fx_n,gx_n))$$

As $n\to\infty d(fz,z)=d(fz,gx_n)= \frac{1}{\alpha}(d(fz,fz)- \frac{\beta}{1+\gamma}d(fz,z))$

Hence $(1-\frac{\beta}{1+\gamma})d(fz,z)\to 0$

So z is a fixed point of f.

Then f and g has the common fixed point z.

Next we have to prove that the fixed point is unique.

Let z and z' are two fixed points, then

$$\begin{aligned} d(z,z') &= d(fz,fz') \\ &\le d(fz,fz') \\ &\le \alpha d(fz,gz)+\frac{\beta}{1+\gamma}d(fz',gz') \\ &\le \alpha d(z',z')+\frac{\beta}{1+\gamma}d(z,z') \\ &\le 0 \end{aligned}$$

That is d(z,z') = 0 which implies z = z'

Therefore the fixed point is unique. □

Definition 3 *Let f and g be two self maps.Two self maps f and g are said to be α- subcompatible if it satisfies*

$d(fx,fy)-d(gx,gy)\le \alpha d(fx,gx)$

Theorem 3.4 *Let (X,d) be a complex valued metric space and*

f,g:X \o (⟶) X be self-mappings and the pair (f,g) are α− subcompatible. Then f and g have a unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary we define a sequence such that

$y_n = fx_n = gx_{n+1}$ for all $n>0$,

Also f and g are α-subcompatible pair

we have d(fx,fy) - d(gx,gy) ≤$\alpha d(fx,gx)$ ......(1)

Let $x=x_{n-1}$ and $y=x_n$

If $y_n=y_{n-1}$ for any n, then $y_n = y_m$ for all $m>n$ hence $\{y_n\}$ is a Cauchy sequence.

If $y_{n-1} \neq y_n$ for all n, then from (1) we have

$$\begin{aligned} d(fx_{n-1},fx_n)-d(gx_{n-1},gx_n) &\leq \alpha d(fx_{n-1},gx_{n-1}) \\ d(fx_{n-1},fx_n) &\leq \alpha d(fx_{n-1},gx_n)+d(gx_{n-1},gx_n) \end{aligned}$$

$$\begin{aligned} d(gx_n,gx_{n+1}) &\leq \alpha d(gx_n,gx_{n+1})+d(gx_{n-1},gx_n) \\ (1-\alpha)d(gx_n,gx_{n+1}) &\leq \alpha d(gx_{n-1},gx_n) \\ d(gx_n,gx_{n+1}) &\leq \frac{\alpha}{1-\alpha}d(gx_{n-1},gx_n) \\ d(gx_n,gx_{n+1}) &\leq hd(gx_{n-1},gx_n) \end{aligned}$$

where $h=\frac{\alpha}{1-\alpha}$. Repeating the process we get

$$\begin{aligned} d(gx_{n+1},gx_n) &\leq hd(gx_n,gx_{n-1}) \\ &\leq h^2d(gx_{n-1},gx_{n-2}) \\ &\leq h^3d(gx_{n-2},gx_{n-3})\ldots \\ &\leq h^n d(gx_0,gx_1) \end{aligned}$$

If m>n we have

$$\begin{aligned} d(gx_n,gx_m) &\leq d(gx_n,gx_{n+1})+d(gx_{n+1},gx_{n+2})+\ldots+d(gx_{m-1},gx_m) \\ &\leq h^n d(gx_0,gx_1)+h^{n+1}d(gx_0,gx_1)+\ldots+h^{m-1}d(gx_0,gx_1) \\ &\leq \frac{h^n}{1-h} \end{aligned}$$

Since $|h|<1, d(gx_n,gx_m)\to 0$ So (gx_n) is a Cauchy sequence. Since complex valued metric space is complete, every Cauchy sequence converges. So it converges to z. That is $\lim_{n\to\infty} gx_n = z$.

We claim that z is a fixed point of g,

For that consider

That is, z is a fixed point of g.

We claim that z is a fixed point of f,

For that consider

Lim d(fz; z)= d(fz; gxn) + d(gxn; z)

=(1/α)d(fxn; gxn)−d(fxn;fxn)+(1/α)d(fxn;fxn)−d(fxn;gxn)

As $n\to\infty, d(fz,z)\to 0$

So z is a fixed point of f.

Hence z is a common fixed point of f and g.

Next we have to prove that the fixed point is unique.

Let z and z'are two fixed points, then

$$\begin{aligned} d(z,z') &= d(fz,fz') \\ &\leq \alpha d(fz',gz')-d(gz,gz') \\ &\leq 0 \end{aligned}$$

$d(fz,fz')\leq 0$

that is fz = fz', hence z = z'

Therefore z is a unique fixed point of f and g. □

Example 5 *Let (X,d) be a complex valued metric space and* $d(z,w)=i|z-w|^2$*, where* $z,w\in X$ *and* $\alpha<1$.

Define $f(x)=\frac{x}{9}$ *and* $g(x)=\frac{8x}{9}$.

Also f(0)=0 and g(0)=0 So that fg(0)=gf(0), {f,g} is α- subcompatible. So that it satisfies all the conditions of the theorem. So that 0 is the unique common fixed point of f and g.

Theorem 3.6 *Let (X,d) be a complex valued metric space. Suppose that the mapping* $f,g:X\to X$ *are weakly reciprocally continous and satisfies* $d(fx,fy)\leq\alpha d(fx,gx)+\frac{\beta}{1+\gamma}d(fy,gy)$ *and* $\frac{\alpha(1+\gamma)}{1+\gamma-\alpha(1+\gamma)-\beta}\leq 1$*. Then f and g have a unique common fixed point.*

Proof: Let $x_0\in X$ be any point in X. we can define a sequence $\{y_n\}$ in X. $y_n=fx_n=gx_{n+1}$ and $d(fx_n,gx_n)<c$

To prove that $\{y_n\}$ is a Cauchy sequence.

Consider

$$\begin{aligned} d(y_n,y_{n+1}) &= d(fx_n,fx_{n+1}) \\ &\leq \alpha d(fx_n,gx_n)+\frac{\beta}{1+\gamma}d(fx_{n+1},gx_{n+1}) \\ &\leq \alpha[d(fx_n,gx_{n+2})+d(gx_{n+2},gx_n]+\frac{\beta}{1+\gamma}d(fx_{n+1},gx_{n+1}) \\ (1-\alpha)d(y_n,y_{n+1}) &\leq \alpha d(y_n,y_{n-1})+\frac{\beta}{1+\gamma}d(y_{n+1},y_n) \\ (1-\alpha)d(y_n,y_{n+1})-\frac{\beta}{1+\gamma}d(y_{n+1},y_n) &\leq \alpha d(y_n,y_{n-1}) \\ &\leq \alpha d(y_n,y_{n-1}) \\ d(y_n,y_{n+1}) &\leq \frac{\alpha(1+\gamma)}{1+\gamma-\alpha(1+\gamma)-\beta}d(y_n,y_{n-1}) \\ d(y_n,y_{n+1}) &\leq hd(y_n,y_{n-1}) \end{aligned}$$

where $h=\frac{\alpha(1+\gamma)}{1+\gamma-\alpha(1+\gamma)-\beta}$

Therefore

$$\begin{aligned} d(y_n,y_{n+1}) &\leq hd(y_n,y_{n-1}) \\ &\leq h^2d(y_{n-2},y_{n-1}) \\ &\leq h^3d(y_{n-2},y_{n-3}).... \\ &\leq h^nd(y_0,y_1) \end{aligned}$$

For all m,n $\in X$

$$d(y_n,y_m) \le d(y_n,y_{n+1})+d(y_{n+1})+.....+d(y_{m-1},y_m)$$
$$(y_n,y_m) \le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+..+d(y_{m-1},y_m)$$
$$\le h^n d(y_0,y_1+h^{n+1}d(y_0,y_1)+..+h^{m-1}d(y_0,y_1$$
$$\le \frac{h^n}{1-h}d(y_0,y_1)$$

Since complex valued metric space is complete, every Cauchy sequence converges. Let it converges to z.
That is $\lim_{n\to\infty} gx_n$ = z.
To prove that z is a fixed point of g,
For that consider $\lim_{n\to\infty} y_n = z$ *We have* $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z$. Given that f and g are weak reciprocal continous,
$\lim_{n\to\infty} fgx_n = fz$ *or* $\lim_{n\to\infty} gfx_n = gz$ As $n\to\infty, d(gz,z)\to 0$
That is z is a fixed point of g. To prove that z is a fixed point of f,
Consider
Lim d(fz; z)= d(fz; gxn) + d(gxn; z)
=(1/α)d(fxn; gxn)−d(fxn;fxn)+(1/α)d(fxn;fxn)−d(fxn;gxn)

As $n\to\infty, d(fz,z)\to 0$
So z is a fixed point of f.

$d(y_n,y_{n+1})\le \alpha d(y_n,y_{n+1})+\frac{\beta}{1+\gamma}d(y_n,y_{n+1})\le 0 d(gz,fz)\le 0$ Hence z is a common fixed point of f and g.

Next we have to prove that fixed point is unique. For that consider two fixed points z and z'.

$d(fz,fz')\le \alpha d(fz,gz)+\frac{\beta}{1+\gamma}d(fz',gz)$ $d(fz',gz)\le 0$

$d(z,z')\le 0$
Therefore z=z'
Hence fixed point is unique. □

Example 7 *Let (X,d) be a complex valued metric space and* $d(z,w)=i|z-w|^2$, *where* $z,w \in X$ *and* $\alpha<1$ *Define* $f(x) = \frac{x}{9}$ *and* $g(x) = \frac{8x}{9}$.
Also f(0)=0 and g(0)=0 So that fg(0)=gf(0),
{f,g} is α subcompatible and reciprocally continous.So that it satisfies all the conditions of the theorem, so that 0 is the unique common fixed point of f and g.

Theorem 3.8 *Let (X,d) be a complex valued metric space and A,B,S and T be self mappings on (X,d). Suppose that the pairs (A,S) and (B,T) are subcompatible and reciprocally continous satisfies*

$$d(Ax,By)-\frac{\alpha}{1+\alpha}d(Sx,Ty) \le \beta\_[d(Sx;By) + d(By; Ty)]+ϒ\, d(Ay; Ty)$$

for all $x,y\in X$ *and* $\alpha,\beta,\gamma\ge 0$. *Then A,B,S and T have a unique common fixed point.*

Proof: Given (A,S) and (B,T) are recipriocally continous
$\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$ whenever there is a sequence $\{x_n\}$ such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $x\in X$

So we have $d(ASx_n,SAx_n)=d(Az,Sz)=0$

So z is a coincidence point of (A,S).

Now we assume that $y_{n+1}=Ax_n=Sx_{n-1}$ and $y_{n+2}=Bx_{n+1}=Tx_n$

Consider $x=x_{n-1}$ and $y=x_n$ then (1) becomes

$$d(Ax_{n-1},Bx_n)-\frac{\alpha}{1+\alpha}d(Sx_{n-1},Tx_n) \leq \beta[d(Sx_{n-1},Bx_n)+d(Bx_n,Tx_n)]+\gamma d(Ax_n Tx_n)$$

If $y_n=y_{n-1}$,we get convergent Cauchy sequence.

If $y_n\neq y_{n-1}$

$$\begin{aligned}
d(Ax_{n-1},Bx_n)-\frac{\alpha}{1+\alpha}d(Sx_{n-1},Tx_n) &\leq \beta[d(Sx_{n-1},Bx_n)+d(Bx_n,Tx_n)] \\
&\quad +\gamma d(Ax_n Tx_n) \\
d(y_n,y_{n+1})-\frac{\alpha}{1+\alpha}d(y_{n+1},y_{n+2}) &\leq \beta[d(y_{n+1},y_{n+1})+d(y_{n+1},y_{n+2})] \\
&\quad +\gamma d(y_{n+1},y_{n+2})
\end{aligned}$$

$$\beta[d(y_{n+1},y_{n+2})]+\gamma d(y_{n+1},y_{n+2})-\frac{\alpha}{1+\alpha}d(y_{n+1},y_{n+2}) \leq -d(y_n,y_{n+1})$$

$$\begin{aligned}
(\beta+\gamma-\frac{\alpha}{1+\alpha})d(y_{n+1},y_{n+2}) &\leq -d(y_n,y_{n+1}) \\
d(y_{n+1},y_{n+2}) &\leq \frac{-1}{(\beta+\gamma-\frac{\alpha}{1+\alpha})}d(y_{n+1},y_n) \\
d(y_{n+1},y_{n+2}) &\leq hd(y_n,y_{n+1})
\end{aligned}$$

where $h=\dfrac{-1}{(\beta+\gamma-\frac{\alpha}{1+\alpha})}$

Therefore

$$\begin{aligned}
d(y_n,y_{n+1}) &\leq hd(y_n,y_{n-1}) \\
&\leq h^2d(y_{n-2},y_{n-1}) \\
&\leq h^3d(y_{n-2},y_{n-3}) \\
&\leq h^nd(y_0,y_1)
\end{aligned}$$

For all $n,m\in N$, with $n<m$

$$\begin{aligned}
d(y_n,y_m) &\leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+\ldots+d(y_{m-1},y_m) \\
&\leq h^nd(y_0,y_1+h^{n+1}d(y_0,y_1)+\ldots+h^{m-1}d(y_0,y_1 \\
&\leq \frac{h^n}{1-h}d(y_0,y_1)
\end{aligned}$$

Since $|h|<1,d(y_n,y_m)\to 0$

So (y_n) is a Cauchy sequence

Since complex valued metric space is complete, every Cauchy sequence converges. So it converges to z.

That islim$_{n\to\infty}y_n=z$.

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} Sx_n = z.$
Next we have to prove that z is a fixed point of A.
To prove that z is a fixed point of A.

$$d(Az,z) = \lim_{n\to\infty} d(Az,Bx_n)$$
$$d(Az,Bx_n) \le \frac{\alpha}{1+\alpha} d(Sz,Tx_n)+\beta[d(Sz,Bx_n)+d(Bx_n,Tx_n)]+\gamma d(Ax_n,Tx_n)$$
$$\le \frac{\alpha}{1+\alpha} d(Sz,Tx_n)+\beta[d(Sz,Bx_n)+d(Bx_n,Tx_n)]+\gamma d(Ax_n,Tx_n)$$

Since (A,S) and (B,T) are subcompatible $d(Az,z)\to 0$
Hence z is a fixed point of A. Since (A,S) is reciprocally continous, z is a fixed point of S also.
Therefore z is a fixed point of A and S.
Next we have to prove that z is a fixed point of B,
For that Consider

$$d(z,Bz) = d(Ax_n,Bz)$$
$$d(Ax_n,Bz) \le$$
$$\frac{\alpha}{1+\alpha} d(Sx_n,Tz)+\beta[d(Sx_n,Bz)+d(Bz,Tz]+\gamma d(Az,Tz)$$

d(Az;Bz)≤(α/1+α)d(Sz;Tz)+β[d(Sz;Bz)+d(Bz;Tz] +ϒd(Az; Tz)

Since (A,S) and (B,T) are subcompatible $d(z,Bz)\to 0$
Hence z is a fixed point of A. Since (B,T) is reciprocally continous z is a fixed point of T also.
Therefore z is a fixed point of B and T.
Hence z is a common fixed point of A,B,S and T.
Next we have to prove that the fixed point is unique.
Let z and w be two fixed points, then

$$d(Az,Bw) \le$$
$$d(Az,Bw) \le$$
$$+\gamma[d(Aw,Sw)+d(Sw,Tw)]$$
$$(1-\frac{\alpha}{1+\alpha}-\beta)d(Az,Bw) \le 0$$

That is $d(z,w)=d(Az,Bw)\le 0$
d(z,w) = 0

Hence fixed point is unique. □

Example 9 *Let (X,d) be a complex valued metric space and* $d(z,w)=i|z-w|$*, where* $z,w \in X$*. Define*

$$A(x)=\begin{cases} 0; x=0 \\ 1/2-x;:0<x\le 1/2 \\ 1+x;:1/2<x\le 1 \end{cases} \quad S(x)=\begin{cases} 0; x=0 \\ 3/2-x;:0<x\le 1/2 \\ x;:1/2<x\le 1 \end{cases}$$

$$B(x)=\begin{cases} 0; x=0 \\ 2:0<x\le 2 \end{cases} \text{and} \quad T(x)=\begin{cases} 0; x=0 \\ 1:0<x\le 2 \end{cases}$$

A,B,S and T satisfies all the conditions of the theorem and have a common fixed point 0 $\in X$

Definition 10 *Let f and S be selfmappings of a complex valued metric space (X,d),the pair (A,S) is said to be weakly subsequentially continous if there exists a sequence(x_n) such that*

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, *for some* $z \in X$ *and*

$\lim_{n\to\infty} ASx_n = Az$ $\lim_{n\to\infty} SAx_n = Sz$

Theorem 3.11 *Let (X,d) be a complex valued metric space and A,B,S and T be self mappings on (X,d).Suppose that the pairs (A,S) and (B,T) are subcompatible and weakly subsequentially continous satisfies*

$$d(Ax,By) - \frac{\alpha}{1+\alpha} d(Sx,Ty) \leq \beta[d(Sx,By)+d(By,Ty)] + \gamma d(Ay,Ty) \quad (2)$$

for all x,y $\in X$.Then A,B,S and T have a common fixed point.

Proof: Since (A,S) is weakly subsequentially continous there exists a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$ $\lim_{n\to\infty} SAx_n = Sz$ Also (A,S) is subcompatible. Let S and T be two selfmaps of a complex valued meric space (X,d), then A and S are said to be subcompatible maps if and only if there exist a sequence x_n in X such that $d(Ax_n,z) = d(Sx_n,z) \leq c$

Consider $x = x_{n-1}$ and $y = x_n$ then (1) becomes

$d(Ax_{n-1},Bx_n) - \frac{\alpha}{1+\alpha} d(Sxn-1,Tx_n) \leq \beta[d(Sx_{n-1},Bx_n)+d(Bx_n,Tx_n)] + \gamma d(Ax_n Tx_n)$

If $y_n = y_{n-1}$,we get convergent Cauchy sequence. If $y_n \neq y_{n-1}$

$$d(Ax_{n-1},Bx_n) - \frac{\alpha}{1+\alpha} d(Sx_{n-1},Tx_n) \leq \beta[d(Sx_{n-1},Bx_n)+d(Bx_n,Tx_n)] + \gamma d(Ax_n Tx_n)$$

$$d(y_n,y_{n+1}) - \frac{\alpha}{1+\alpha} d(y_{n+1},y_{n+2}) \leq \beta[d(y_{n+1},y_{n+1})+d(y_{n+1},y_{n+2})] + \gamma d(y_{n+1},y_{n+2})$$

$$\beta[d(y_{n+1},y_{n+2})] + \gamma d(y_{n+1},y_{n+2}) - \frac{\alpha}{1+\alpha} d(y_{n+1},y_{n+2}) \leq -d(y_n,y_{n+1})$$

$$(\beta+\gamma-\frac{\alpha}{1+\alpha}) d(y_{n+1},y_{n+2}) \leq -d(y_n,y_{n+1})$$

$$d(y_{n+1},y_{n+2}) \leq \frac{-1}{(\beta+\gamma-\frac{\alpha}{1+\alpha})} d(y_{n+1},y_n)$$

$$d(y_{n+1},y_{n+2}) \leq hd(y_n,y_{n+1})$$

where $h = \frac{-1}{(\beta+\gamma-\frac{\alpha}{1+\alpha})}$

$$\text{Therefore } d(y_n,y_{n+1}) \leq hd(y_n,y_{n-1})$$

$$\begin{aligned} &\leq h^2 d(y_{n-2},y_{n-1}) \\ &\leq h^3 d(y_{n-2},y_{n-3}) \\ &\leq h^n d(y_0,y_1) \end{aligned}$$

For all $n,m \in N$, with $n<m$

$$\begin{aligned} d(y_n,y_m) &\leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+\ldots+d(y_{m-1},y_m) \\ &\leq h^n d(y_0,y_1)+h^{n+1}d(y_0,y_1)+\ldots+h^{m-1}d(y_0,y_1) \\ &\leq \frac{h^n}{1-h} d(y_0,y_1) \end{aligned}$$

Since $|h|<1, d(y_n,y_m)\to 0$

So (y_n) is a Cauchy sequence

Since complex valued metric space is complete, every Cauchy sequence converges. So it converges to z.

That is$\lim_{n\to\infty} y_n = z$.

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} Sx_n = z$.

Next we have to prove that z is a fixed point A.

To prove that z is a fixed point of A.

$$\begin{aligned} d(Az,z) &= \lim_{n\to\infty} d(Az,Bx_n) \\ d(Az,Bx_n) &\leq \frac{\alpha}{1+\alpha} d(Sz,Tx_n)+\beta[d(Sz,Bx_n)+d(Bx_n,Txn)]+\gamma d(Ax_n,Tx_n) \\ &\leq \frac{\alpha}{1+\alpha} d(Sz,Tx_n)+\beta[d(Sz,Bx_n)+d(Bx_n,Tx_n)]+\gamma d(Ax_n,Tx_n) \end{aligned}$$

Since (A,S) and (B,T) are subcompatible $d(Az,z)\to 0$. Hence z is a fixed point of A. Since (A,S) is weakly subsequentially continous z is a fixed point of S also. Therefore z is a fixed point of A and S.

Next we have to prove that z is a fixed point of B,

For that consider

$$\begin{aligned} d(z,Bz) &= d(Ax_n,Bz) \\ d(Ax_n,Bz) &\leq \end{aligned}$$

$$\frac{\alpha}{1+\alpha} d(Sx_n,Tz)+\beta[d(Sx_n,Bz)+d(Bz,Tz)]+\gamma d(Az,Tz)$$

Since (A,S) and (B,T) are subcompatible $d(z,Bz)\to 0$ Hence z is a fixed point of A.Since (B,T) is weakly subsequentially continous z is a fixed point of T also. Therefore z is a fixed point of B and T. Hence z is a common fixed point of A,B,S and T.

Next we have to prove that the fixed point is unique.

Let z and w be two fixed points

d(Az;Bz) ≤ (α/1+α)d(Sz; Tz) +→β[d(Sz;Bz) + d(Bz; Tz] +ϒd(Az; Tz)

$$d(Az,Bw) \leq$$

$$+\gamma[d(Aw,Sw)+d(Sw,Tw)]$$
$$\leq \quad 0$$

That is $d(z,w)=d(Az,Bw)\leq 0$

$d(z,w)=0$

Hence fixed point is unique. □$(1-\frac{\alpha}{1+\alpha}-\beta)d(Az,Bw)$

Example 12

Let (X,d) be a complex valued metric space and d(z,w)= i|z−w|,where z,w ∈X.Define

$$A(x)=\begin{cases}0; x=0\\ 1/2-x;:0<x\leq 1/2\\ 1+x;:1/2<x\leq 1\end{cases} \quad S(X)=\begin{cases}0; x=0\\ 1/2-x;:0<x\leq 1/2\\ \frac{3-2x}{2};:1/2<x\leq 1\end{cases}$$

‘

$$B(X)=\begin{cases}0; x=0\\ 1/3;:0<x\leq 1\end{cases} \text{and} \quad T(x)=\begin{cases}0; x=0\\ 1;:0<x\leq 1\end{cases}$$

A,L,M and S satisfies all the conditions of the theorem and have a unique common fixed point 0 ∈X. In this example L and A commute at their coincidence point 0 ∈ X. So A and S are weakly subsequentially continous. Similarly B and T are weakly subsequentially continous.

Theorem 3.13 *Let (X,d) be a complex valued metric space and A,B,S T be self mappings on (X,d).Suppose that the pairs (A,S) and (B,T) are subcompatible and reciprocally continuous satisfies*

$$d(Ax;By) - (\alpha/1+\alpha)d(Sx,Ty)\leq \beta[d(Sx;By) + d(By; Ty)] +\Upsilon d(Ay; Ty) \qquad (1)$$

for all x,y∈X and α,β,γ≥0.Then A,B,S and T have a unique common fixed point.

Proof: Given (A,S) and (B,T) are weakly recipriocally continous $\lim_{n\to\infty} ASx_n=Az$ and $\lim_{n\to\infty} SAx_n=Sz$ whenever there is a sequence $\{x_n\}$ such that

$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=z$ for some $x\in X$

So we have $d(ASx_n,SAx_n)=d(Az,Sz)=0$

So z is a coincidence point of (A,S).

Now we assume that $y_{n+1}=Ax_n=Sx_{n-1}$ and $y_{n+2}=Bx_{n+1}=Tx_n$

Consider $x=x_{n-1}$ and $y=x_n$ then (1) becomes

$$d(Ax_{n-1},Bx_n)-\frac{\alpha}{1+\alpha}d(Sx_{n-1},Tx_n) \quad \leq \quad \beta[d(Sx_{n-1},Bx_n)+d(Bx_n,Tx_n)]+\gamma d(Ax_nTx_n)$$

If$y_n=y_{n-1}$,we get convergent Cauchy sequence. If$y_n\neq y_{n-1}$

$$d(Ax_{n-1},Bx_n)-\frac{\alpha}{1+\alpha}d(Sxn-1,Tx_n) \quad \leq \beta[d(Sx_{n-1},Bx_n)+d(Bx_n,Tx_n)]$$
$$+\gamma d(Ax_nTx_n)\ d(y_n,y_{n+1})$$
$$-\frac{\alpha}{1+\alpha}d(y_{n+1},y_{n+2})$$
$$\leq \quad \beta[d(y_{n+1},y_{n+1})+d(y_{n+1},y_{n+2})]$$

$$+\gamma d(y_{n+1},y_{n+2})$$

$$\beta[d(y_{n+1},y_{n+2})]+\gamma d(y_{n+1},y_{n+2})-\frac{\alpha}{1+\alpha}d(y_{n+1},y_{n+2}) \leq -d(y_n,y_{n+1})$$

$$(\beta+\gamma-\frac{\alpha}{1+\alpha})d(y_{n+1},y_{n+2}) \leq -d(y_n,y_{n+1})$$

$$d(y_{n+1},y_{n+2}) \leq \frac{-1}{(\beta+\gamma-\frac{\alpha}{1+\alpha})}d(y_{n+1},y_n)$$

$$d(y_{n+1},y_{n+2}) \leq hd(y_n,y_{n+1})$$

where $h=\frac{-1}{(\beta+\gamma-\frac{\alpha}{1+\alpha})}$

$$\begin{aligned} \textit{Therefore } d(y_n,y_{n+1}) &\leq hd(y_n,y_{n-1}) \\ &\leq h^2d(y_{n-2},y_{n-1}) \\ &\leq h^3d(y_{n-2},y_{n-3}) \\ &\leq h^nd(y_0,y_1) \end{aligned}$$

For all $n,m\in N$, with $n<m$

$$\begin{aligned} d(y_n,y_m) &\leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+...+d(y_{m-1},y_m) \\ &\leq h^nd(y_0,y_1+h^{n+1}d(y_0,y_1)+...+h^{m-1}d(y_0,y_1 \\ &\leq \frac{h^n}{1-h}d(y_0,y_1) \end{aligned}$$

Since $|h|<1,d(y_n,y_m)\to 0$

So (y_n) is a Cauchy sequence

Since complex valued metric space is complete,every Cauchy sequence converges.So it converges to z. That is$\lim_{n\to\infty}y_n=z.$

$\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}gx_{n+1}=\lim_{n\to\infty}Sx_n=z.$

Next we have to prove that z is a fixed point of A.

To prove that z is a fixed point of A.

$$d(Az,z) = \lim_{n\to\infty}d(Az,Bx_n)$$

$$\begin{aligned} d(Az,Bx_n) &\leq \frac{\alpha}{1+\alpha}d(Sz,Tx_n)+\beta[d(Sz,Bx_n)+d(Bx_n,Txn)]+\gamma d(Ax_n,Tx_n) \\ &\leq \frac{\alpha}{1+\alpha}d(Sz,Tx_n)+\beta[d(Sz,Bx_n)+d(Bx_n,Tx_n)]+\gamma d(Ax_n,Tx_n) \end{aligned}$$

Since (A,S) and (B,T) are subcompatible $d(Az,z)\to 0$ Hence z is a fixed point of A.Since (A,S) is weakly subsequentially continous z is a fixed point of S also

Therefore z is a fixed point of A and S.

Next we have to prove that z is a fixed point of B,

For that consider

$$d(z,Bz) = d(Ax_n,Bz)$$

$$d(Ax_n,B)z \leq \mathrm{d(Az,Bz)}$$

Since (A,S) and (B,T) are subcompatible $d(z,Bz)\to 0$
Hence z is a fixed point of A.Since (B,T) is weakly subsequentially continous z is a fixed point of T also.
Therefore z is a fixed point of B and T.
Hence z is a common fixed point of A,B,S and T.
Next we have to prove that the fixed point is unique.
Let z and w be two fixed points, then

$$d(Az;Bw) \le (\alpha/1+\alpha)d(Sz;\, Tw)+\beta[d(Sw;Bw) + d(Bz;\, Tz)] +\Upsilon d(Az;\, Tz)$$

$$(1-\frac{\alpha}{1+\alpha}-\beta)d(Az,Bw) \le 0$$

That is d(z,w)=d(Az,Bw)≤0
d(z,w)=0 Hence fixed point is unique. □

Example 14

Let (X,d) be a complex valued metric space and $d(z,w) = i|z-w|$,where $z,w\in X$.Define

$$A(x)=\begin{cases} 0; x=0 \\ 1/4-x;:0<x\le 1/4 \\ 1+x;:1/4<x\le 1 \end{cases} \quad S(X)=\begin{cases} 0; x=0 \\ 1/2-x;:0<x\le 1/4 \\ \frac{2-x}{4};:1/4<x\le 1 \end{cases}$$

$$B(X)=\begin{cases} 0; x=0 \\ 1/2;:0<x\le 1 \end{cases} \text{and} \quad T(x)=\begin{cases} 0; x=0 \\ 1;:0<x\le 1 \end{cases}$$

A,L,M and S satisfies all the conditions of the theorem and have a unique common fixed point $0 \in X$ .In this example A and S commute at their coincidence point $0 \in X$ .So A and S are subcompatible

Similarly B and T are subcompatible.

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