Oscillation of Third Order Sub linear Neutral Delay Difference Equations

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Abstract :This paper discusses oscillatory behavior of the third order difference equations with sub linear neutral term.

IndexTerms **–oscillation, third order, sub linear, difference equation.**

I. Introduction

Recent decade witnessed an increasing interest in the study of oscillation of third-order neutral delay differential and difference equations[1, 2, 3]. Difference equations have found useful applications in population dynamics, bifurcation analysis, circuit theory, dynamic behavior of delayed network systems[7]. Oscillation of solutions of difference equations is one of the important qualitative properties[4, 5, 6]. In this paper, we consider the following third order neutral difference equations of the form

$$\Delta[a(n)\Delta[b(n)\Delta z(n)]] + q(n)x(n-\delta) = 0, n \geq n_0 \tag{1}$$

Here $z(n) = [x(n) + p(n)x^{\alpha}(n-\tau)]$. Assume that the following conditions hold;

(C_1) $0 < \alpha \leq 1$ is the ratio of odd positive integers;

(C_2) $a(n), b(n), p(n), q(n)$ are all positive sequences;

(C_3) $\tau(n), \delta(n)$ are all positive such that $\tau(n) \geq n; \delta(n) \geq n;$

(C_4) $\lim_{n\to\infty} \tau(n) = \lim_{n\to\infty} \delta(n) = \infty$.

Also we assume

(A) $\sum_{n=n_0}^{\infty} \frac{1}{a(n)} = \infty; \sum_{n=n_0}^{\infty} \frac{1}{b(n)} = \infty.$

(B) $\sum_{n=n_0}^{\infty} \frac{1}{a(n)} < \infty; \sum_{n=n_0}^{\infty} \frac{1}{b(n)} = \infty.$

A non-trivial solution $x(n)$ is said to be oscillatory if it is neither eventually positive or eventually negative; otherwise, it is non oscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

II. Oscillation results

Our objective in this paper is to present some sufficient conditions for the oscillation of solutions of equation (1).

Theorem 2.1. Assume $(C_1)-(C_4)$ and (B) holds, $0 \leq p(n) \leq p_1 \leq 1$. If there exists a positive function ϕ, such that for all sufficiently large $n_3 > n_2 > n_1 \geq n_0$, we have

$$\lim_{n\to\infty} Sup \sum_{m=n_3}^{n-1} \left[q(m)\phi(m)p_*(m-\delta) \frac{\sum_{j=n_2}^{m-\delta-1} \frac{1}{b(j)} \sum_{i=n_2}^{j-1} \frac{1}{a(i)}}{\sum_{i=n_1}^{m-1} \frac{1}{a(i)}} - \frac{a(m)\Delta^2\phi(m)}{4\phi(m)} \right] = \infty \tag{2}$$

Where $p_*(n) = 1 - \frac{p(n)}{M^{1-\alpha}}$, hold for all constants $M > 0$, then any solution $x(n)$ of (1) is oscillatory.

Proof. Suppose $x(n)$ is positive solution of (1). By condition (A), there exists a possible case such that $z(n) > 0, \Delta z(n) > 0, \Delta[b(n)\Delta z(n)] > 0, \Delta[a(n)\Delta[b(n)\Delta z(n)]] < 0.$ Assume that $z(n)$ satisfying the case, there exists

$n \geq n_1$ such that $z(n) > 0$; $z(n-\delta) > 0$ $\Delta z(n) > 0$ then $z(n)$ considered monotonically increasing, there exists a constant $M > 0$, such that $z(n) \geq M$. Now from the definition of $z(n)$ we have

$$x(n) = z(n) - p(n)x^{\alpha}(n-\tau) \geq z(n) - p(n)z^{\alpha}(n-\tau) \geq \left[1 - p(n)\frac{1}{M^{1-\alpha}}\right]z(n) \geq p_*(n)z(n) \quad (3)$$

Where $p_*(n) = \left[1 - p(n)\frac{1}{M^{1-\alpha}}\right]$. Since,

$$b(n)\Delta z(n) \geq \sum_{m=n_1}^{n-1} \frac{a(m)\Delta[b(m)\Delta z(m)]}{a(m)}$$

$$\frac{b(n)\Delta z(n)}{\sum_{m=n_1}^{n-1} \frac{1}{a(m)}} \geq a(n)\Delta[b(n)\Delta z(n)] \quad (4)$$

We have

$$\Delta\left[\frac{b(n)\Delta z(n)}{\sum_{m=n_1}^{n-1} \frac{1}{a(m)}}\right] \leq 0 \quad (5)$$

Now, $\sum_{m=n_2}^{n-1} \Delta z(m) = z(n) - z(n_2)$ yields

$$z(n) = z(n_2) + \sum_{m=n_2}^{n-1} \Delta z(m) = z(n_2) + \sum_{m=n_2}^{n-1} \frac{b(m)\Delta z(m)\sum_{i=n_1}^{m-1}\frac{1}{a(i)}}{b(m)\sum_{i=n_1}^{m-1}\frac{1}{a(i)}}$$

$$\geq \frac{b(n)\Delta z(n)}{\sum_{i=n_1}^{n-1}\frac{1}{a(i)}} \sum_{m=n_2}^{n-1}\left[\frac{1}{b(m)}\sum_{i=n_1}^{m-1}\frac{1}{a(i)}\right]$$

$$\frac{z(n)}{b(n)\Delta z(n)} \geq \frac{\sum_{m=n_2}^{n-1}\left[\frac{1}{b(m)}\sum_{i=n_1}^{m-1}\frac{1}{a(i)}\right]}{\sum_{i=n_1}^{n-1}\frac{1}{a(i)}} \quad (6)$$

Let us define the Reccati transformation,

$$w(n) = \phi(n)\frac{a(n)\Delta[b(n)\Delta z(n)]}{b(n)\Delta z(n)}, n \geq n_1. \quad (7)$$

Notice that $w(n) > 0$ for $n \geq n_2 \geq n_1$. Now

$$\Delta(w(n)) = \phi(n)\frac{\Delta[a(n)\Delta[b(n)\Delta z(n)]]}{b(n)\Delta z(n)} + \Delta\phi(n)\frac{a(n+1)\Delta[b(n+1)\Delta z(n+1)]}{b(n+1)\Delta z(n+1)}$$
$$-\phi(n)\frac{a(n+1)\Delta[b(n+1)\Delta z(n+1)]\{b(n+1)\Delta z(n+1) - b(n)\Delta z(n)\}}{b(n)\Delta z(n)b(n+1)\Delta z(n+1)}$$

It follows from (1) and (7)

$$\Delta(w(n)) = \frac{-q(n)\varphi(n)x(n-\delta)}{b(n)\Delta z(n)} + \Delta\varphi(n)\frac{w(n+1)}{\varphi(n+1)} - \frac{\varphi(n)}{a(n)}\frac{w(n+1)}{\varphi(n+1)}\frac{a(n)\Delta[b(n)\Delta z(n)]}{b(n)\Delta z(n)}$$

$a(n)\Delta[b(n)\Delta z(n)]$ is non increasing and $[b(n)\Delta z(n)]$ is increasing. From (3)

$$\Delta(w(n)) \le \Delta\phi(n)\frac{w(n+1)}{\phi(n+1)} - \frac{\phi(n)}{a(n)}\frac{w^2(n+1)}{\phi^2(n+1)} - q(n)\phi(n)\frac{x(n-\delta)}{b(n)\Delta z(n)}$$

$$\le -\left[\frac{\phi(n)}{a(n)}\frac{w^2(n+1)}{\phi^2(n+1)} - \Delta\phi(n)\frac{w(n+1)}{\phi(n+1)}\right] - q(n)\phi(n)p_*(n-\delta)\frac{z(n-\delta)}{b(n)\Delta z(n)}$$

$$\le -\left[\frac{\varphi(n)}{a(n)}\frac{w^2(n+1)}{\varphi^2(n+1)} - \Delta\varphi(n)\frac{w(n+1)}{\varphi(n+1)}\right] - q(n)\varphi(n)p_*(n-\delta)\frac{\sum_{m=n_2}^{n-\delta-1}\left[\frac{1}{b(m)}\sum_{i=n_1}^{m-1}\frac{1}{a(i)}\right]}{\sum_{i=n_1}^{n-1}\frac{1}{a(i)}}$$

$$\le -\left[\frac{\sqrt{\varphi(n)}}{\sqrt{a(n)}}\frac{w(n+1)}{\varphi(n+1)} - \frac{\Delta\varphi(n)}{2}\frac{\sqrt{a(n)}}{\sqrt{\varphi(n)}}\right]^2 + \frac{\Delta^2\varphi(n)a(n)}{4\varphi(n)} - q(n)\varphi(n)p_*(n-\delta)\frac{\sum_{m=n_2}^{n-\delta-1}\left[\frac{1}{b(m)}\sum_{i=n_1}^{m-1}\frac{1}{a(i)}\right]}{\sum_{i=n_1}^{n-1}\frac{1}{a(i)}}$$

Which implies,

$$\Delta(w(n)) \le -q(n)\varphi(n)p_*(n-\delta)\frac{\sum_{m=n_2}^{n-\delta-1}\left[\frac{1}{b(m)}\sum_{i=n_1}^{m-1}\frac{1}{a(i)}\right]}{\sum_{i=n_1}^{n-1}\frac{1}{a(i)}} + \frac{\Delta^2\varphi(n)a(n)}{4\varphi(n)}$$

Summing from $n_3(>n_2)$ to $n-1$, we get

$$w(n)-w(n_3) \le \sum_{m=n_3}^{n-1}\left[\frac{\Delta^2\phi(m)a(m)}{4\phi(m)} - q(m)\phi(m)p_*(m-\delta)\frac{\sum_{j=n_2}^{m-\delta-1}\left[\frac{1}{b(j)}\sum_{i=n_1}^{j-1}\frac{1}{a(i)}\right]}{\sum_{i=n_1}^{m-1}\frac{1}{a(i)}}\right]$$

$$\sum_{m=n_3}^{n-1}\left[q(m)\phi(m)p_*(m-\delta)\frac{\sum_{j=n_2}^{m-\delta-1}\left[\frac{1}{b(j)}\sum_{i=n_1}^{j-1}\frac{1}{a(i)}\right]}{\sum_{i=n_1}^{m-1}\frac{1}{a(i)}} - \frac{\Delta^2\phi(m)a(m)}{4\phi(m)}\right] \le w(n_3)$$

Letting $n \to \infty$

$$\lim_{n\to\infty} Sup\sum_{m=n_3}^{n-1}\left[q(m)\phi(m)p_*(m-\delta)\frac{\sum_{j=n_2}^{m-\delta-1}\left[\frac{1}{b(j)}\sum_{i=n_1}^{j-1}\frac{1}{a(i)}\right]}{\sum_{i=n_1}^{m-1}\frac{1}{a(i)}} - \frac{\Delta^2\phi(m)a(m)}{4\phi(m)}\right] \le w(n_3)$$

Which contradicts (2). Hence complete the proof.

Theorem 2.2. Assume $(C_1)-(C_4)$ and (B) holds and

$$\sum_{j=n_0}^{\infty}\frac{1}{b(j)}\sum_{i=j}^{\infty}\frac{1}{a(i)}\sum_{m=i}^{\infty}q(m) = \infty \qquad (8)$$

Then any solution $x(n)$ of (1) is either oscillatory or $\lim_{n\to\infty} x(n) = 0$.

Proof. Suppose $x(n)$ is positive solution of (1). By condition (A), there exists a possible case such that $z(n) > 0, \Delta z(n) < 0, \Delta[b(n)\Delta z(n)] > 0, \Delta[a(n)\Delta[b(n)\Delta z(n)]] < 0.$ Since $z(n) > 0$ and $\Delta z(n) < 0$ there exist a finite limit, $\lim_{n\to\infty} z(n) = l$.

We shall prove that $l=0$. Assume that $l>0$. Then for any $\varepsilon>0$, we have $l+\varepsilon>z(n)>l$; choose $0<\varepsilon<\frac{l(1-p)}{p}$. From the definition of $z(n)$, we obtain

$$x(n)=z(n)-p(n)x^{\alpha}(n-\tau)>l-pz(n-\delta)>1-p(l+\varepsilon)=k(l+\varepsilon)>kz(n)$$

Where $k=\frac{l-p(l+\varepsilon)}{(l+\varepsilon)}>0$

From (1),

$$\Delta[a(n)\Delta[b(n)\Delta z(n)]]=-q(n)x(n-\delta)\leq -kq(n)z(n-\delta)$$

Summing from n to ∞, we get

$$\sum_{m=n}^{\infty}\Delta[a(m)\Delta[b(m)\Delta z(m)]]\leq -k\sum_{m=n}^{\infty}q(m)z(m-\delta)$$

$$\frac{kl}{a(n)}\sum_{m=n}^{\infty}q(m)\leq \Delta[b(n)\Delta z(n)]$$

Summing again from n_1 to ∞, we get

$$\sum_{i=n_1}^{\infty}\left[\frac{kl}{a(i)}\sum_{m=i}^{\infty}q(m)\right]\leq \sum_{i=n_1}^{\infty}\Delta[b(i)\Delta z(i)]$$

$$\frac{kl}{b(n_1)}\sum_{i=n_1}^{\infty}\frac{1}{a(i)}\sum_{m=i}^{\infty}q(m)\leq -\Delta z(n_1)$$

Again taking summing from n_0 to ∞

$$kl\left[\sum_{j=n_0}^{\infty}\frac{1}{b(j)}\sum_{i=j}^{\infty}\frac{1}{a(i)}\sum_{m=i}^{\infty}q(m)\right]\leq -\sum_{j=n_0}^{\infty}\Delta z(j)$$

$$\sum_{j=n_0}^{\infty}\frac{1}{b(j)}\sum_{i=j}^{\infty}\frac{1}{a(i)}\sum_{m=i}^{\infty}q(m)\leq \frac{z(n_0)}{kl}$$

Which is a contradiction to (8). Hence we get the required result.

Theorem 2.3. Assume $(C_1)-(C_4)$ and (A) holds, $0\leq p(n)\leq p_1\leq 1$. If there exists a positive function ψ, such that for all sufficiently large $n_2>n_1\geq n_0$, we have

$$\lim_{n\to\infty} Sup\sum_{m=n_2}^{n-1}\left[q(m)\psi(m)p_*(m)\sum_{i=n_1}^{m-\delta-1}\frac{1}{b(i)}-\frac{1}{4a(m)\psi(m)}\right]=\infty \qquad (9)$$

Where $p_*(n)=1-\frac{p(n)}{M^{1-\alpha}}$ and $\psi(n)=\sum_{m=n}^{\infty}\frac{1}{a(m)}$ hold for all constants $M>0$, then any solution $x(n)$ of (1) is oscillatory.

Proof. Suppose $x(n)$ is positive solution of (1). By condition (B), there exists a possible case such that $z(n)>0,\Delta z(n)>0,\Delta[b(n)\Delta z(n)]<0,\Delta[a(n)\Delta[b(n)\Delta z(n)]]<0.$

Hence $a(n)\Delta[b(n)\Delta z(n)]$ is non increasing,

$$a(m)\Delta[b(m)\Delta z(m)]\leq a(n)\Delta[b(n)\Delta z(n)],\ \ m>n\geq n_1$$

Both side dividing $a(m)$ and summing from n to $l-1$, we get

$$\sum_{m=n}^{l-1}\Delta[b(m)\Delta z(m)]\leq a(n)\Delta[b(n)\Delta z(n)]\sum_{m=n}^{l-1}\frac{1}{a(m)}$$

$$b(l)\Delta z(l)\leq b(n)\Delta z(n)+a(n)\Delta[b(n)\Delta z(n)]\sum_{m=n}^{l-1}\frac{1}{a(m)}$$

Letting $l\to\infty$

$$\lim_{l\to\infty} b(l)\Delta z(l) \le b(n)\Delta z(n) + a(n)\Delta[b(n)\Delta z(n)]\lim_{l\to\infty}\sum_{m=n}^{l-1}\frac{1}{a(m)}$$

$$0 \le b(n)\Delta z(n) + a(n)\Delta[b(n)\Delta z(n)]\sum_{m=n}^{\infty}\frac{1}{a(m)}$$

$$-\frac{a(n)\Delta[b(n)\Delta z(n)]\sum_{m=n}^{\infty}\frac{1}{a(m)}}{b(n)\Delta z(n)} \le 1$$

Now define $\phi(n)$ as follows,

$$\phi(n) = \frac{a(n)\Delta[b(n)\Delta z(n)]}{b(n)\Delta z(n)}, n \ge n_1 \tag{10}$$

Notice that $\phi(n) < 0$ for $n \ge n_1$, Thus

$$-\phi(n)\sum_{m=n}^{\infty}\frac{1}{a(m)} \le 1$$

$$-\phi(n)\psi(n) \le 1 \text{ where } \psi(n) = \sum_{m=n}^{\infty}\frac{1}{a(m)}$$

From (10), we get

$$\Delta\phi(n) = \Delta\left[\frac{a(n)\Delta[b(n)\Delta z(n)]}{b(n)\Delta z(n)}\right]$$

$$= \frac{[b(n)\Delta z(n)]\Delta[a(n)\Delta[b(n)\Delta z(n)]]}{[b(n)\Delta z(n)][b(n+1)\Delta z(n+1)]} - \frac{a(n)\Delta[b(n)\Delta z(n)]\Delta[b(n)\Delta z(n)]}{[b(n)\Delta z(n)][b(n+1)\Delta z(n+1)]}$$

$$= \frac{\Delta[a(n)\Delta[b(n)\Delta z(n)]]}{b(n+1)\Delta z(n+1)} - \frac{a(n)\Delta^2[b(n)\Delta z(n)]}{[b(n)\Delta z(n)][b(n+1)\Delta z(n+1)]}$$

$$= \frac{-q(n)x(n-\delta)}{b(n)\Delta z(n)} - \frac{a(n)\Delta^2[b(n)\Delta z(n)]}{[b(n)\Delta z(n)]^2}$$

$$= \frac{-q(n)x(n-\delta)}{b(n)\Delta z(n)} - \frac{\phi^2(n)}{a(n)}$$

In view of case;

$$z(n) \ge b(n)\sum_{m=n_1}^{n-1}\frac{1}{b(m)}\Delta z(n)$$

$$\frac{z(n)}{b(n)\Delta z(n)} \ge \sum_{m=n_1}^{n-1}\frac{1}{b(m)} \tag{11}$$

Hence, $\Delta\left[\frac{z(n)}{\sum_{m=n_1}^{n-1}\frac{1}{b(m)}}\right] \le 0$, which implies that $\frac{z(n-\delta)}{z(n)} \ge \frac{\sum_{m=n_1}^{n-\delta-1}\frac{1}{b(m)}}{\sum_{m=n_1}^{n-1}\frac{1}{b(m)}}$. Now apply (3) and (11) to $\Delta\phi(n)$, we have

$$\Delta\phi(n) \le \frac{-q(n)p_*(n)z(n-\delta)}{b(n)\Delta z(n)} - \frac{\phi^2(n)}{a(n)}$$

$$\le -q(n)p_*(n)\sum_{m=n_1}^{n-\delta-1}\frac{1}{b(m)} - \frac{\varphi^2(n)}{a(n)}$$

Thus multiplying by $\psi(n)$ and summing it from $n_2(>n_1)$ to $n-1$

$$\sum_{m=n_2}^{n-1}\Delta\phi(m)\psi(m)\le-\sum_{m=n_2}^{n-1}\left[q(m)p_*(m)\psi(m)\sum_{i=n_1}^{m-\delta-1}\frac{1}{b(i)}+\sum_{m=n_2}^{n-1}\left[\frac{\phi^2(m)}{a(m)}\psi(m)+\frac{\phi(m)}{a(m)}\right]\right]$$

$$\sum_{m=n_2}^{n-1}q(m)p_*(m)\psi(m)\sum_{i=n_1}^{m-\delta-1}\frac{1}{b(i)}+\sum_{m=n_2}^{n-1}\left[\frac{\phi^2(m)}{a(m)}\psi(m)+\frac{\phi(m)}{a(m)}\right]\le-\phi(n)\psi(n)+\phi(n_2)\psi(n_2)$$

Since $-\phi(n)\psi(n)\le 1$

$$\sum_{m=n_2}^{n-1}q(m)p_*(m)\psi(m)\sum_{i=n_1}^{m-\delta-1}\frac{1}{b(i)}+\sum_{m=n_2}^{n-1}\left[\frac{\sqrt{\psi(m)}}{\sqrt{a(m)}}\phi(m)-\frac{1}{2\sqrt{a(m)\psi(m)}}\right]^2-\sum_{m=n_2}^{n-1}\frac{1}{4a(m)\psi(m)}\le 1+\phi(n_2)\psi(n_2)$$

$$\sum_{m=n_2}^{n-1}\left[q(m)p_*(m)\psi(m)\sum_{i=n_1}^{m-\delta-1}\frac{1}{b(i)}-\frac{1}{4a(m)\psi(m)}\right]\le 1+\phi(n_2)\psi(n_2)$$

Letting $n\to\infty$

$$\lim_{n\to\infty}Sup\sum_{m=n_2}^{n-1}\left[q(m)p_*(m)\psi(m)\sum_{i=n_1}^{m-\delta-1}\frac{1}{b(i)}-\frac{1}{4a(m)\psi(m)}\right]\le 1+\phi(n_2)\psi(n_2)$$

Which contradicts (9) and the proof is complete.

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