L (δ^*-OPEN, OPEN) MAPPINGS

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Abstract: By using δ^*-open sets of Singal and Yadav [**14**], we introduce the concept of L (δ^*-open, open) mappings, named as nomenclature of mappings mentioned in paper [**5**]. This new class of L (δ^*-open, open) mappings is a super class of strongly continuous mappings of Levine [**6**] and a subclass of the class of strongly θ-continuous mappings of Noiri [**11**] as well as that of super continuous mappings of Munshi and Bassan [**10**]. L (δ*-open, open) mappings are shown as independent of strongly semicontinuous mappings due to yadav [**16**]. Various characterizations and some preservation properties of the new mappings are investigated.

Key words: δ-open, δ^*-open sets; strongly continuous, strongly semcontinuous, L (δ^*-open, open), super continuous and strongly θ-continuous mappings.
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1. Introduction

In 1960, Levine [**6**] introduced the concept of strongly continuous mappings in topological spaces and obtained their properties. In 1963, Levine [**7**] introduced the notion of semi-open sets as a generalization of open sets. In 1966, Velico [**13**] introduced the concepts of θ-open and δ-open sets and obtained their properties. In 1974, Arya and Gupta [**1**] introduced completely continuous mappings and obtained their characterizations. In 1980, by using θ-open sets, Noiri [**11**] introduced strongly θ-continuous mappings. In 1981, Jain [**4**] introduced the concepts of totally continuous mappings. In 1982, Munshi and Bassan [**10**] introduced the notion of super continuous mappings. In 1987, Singal and Yadav [**14**] introduced a weak form of δ-open set is called δ^*-open set. In 1988, Yadav [**16**] introduced the notion of strongly semi-continuous mappings. In 1990, Popa [**12**] introduced the concept of almost feebly continuous functions and obtained their properties.

2. Preliminaries

A subset G of a space X is defined as **δ-open** [**13**] if for each x ∈ G, there exists a regular open set H such that x ∈ H ⊂ G. Similarly, Singal and Yadav [**14**] defined G ⊂ X to be **δ^*-open** if for each x ∈ G, there exists a clopen set H such that x ∈ H ⊂ G. A set A is **δ^*-closed** (resp. **δ-closed**) iff X - A is δ^*-open (resp. δ-open) and (**δ^*-clopen** if it is both δ^*-open and δ^*-closed). The collection of all δ^*-open sets in a space (X, ℑ), denoted by **δ^*-O(X, ℑ)**, is a topology ℑ* on X, called **O-dimensionalization** [**14**]of ℑ. They further showed that ℑ = ℑ* iff the space $(X, \Im)$ is a O-dimensional space. $A \subset X$ is said to be **semiopen** [**7**] (resp. **feebly-open** [**9**]) if $G \subset A \subset cl(G)$, (resp. $G \subset A \subset s\text{-}cl(G)$, where s-cl denotes semiclosure) for some open set G. A is called **θ-open** [**13**] if for each $x \in A$, there exists an open set G such that $x \in G \subset cl(G) \subset A$. Obviously, every δ^*-open set is δ-open as well as θ-open. However, in $X = \{a, b, c\}$ with $\Im = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then $A = \{a\}$ is δ-open but not δ^*-open and in the usual space of reals an open interval (a, b) is θ-open but not δ^*-open. The smallest δ^*-closed set containing A is called **δ^*-closure** of A denoted by **δ^*-cl(A)**, and the largest δ^*-open set in A is called **δ^*-interior** of A, denoted by **δ^*-int(A)**. A is δ^*-closed (resp. δ^*-open) iff $A = \delta^*\text{-}cl(A)$ (resp. $A = \delta^*\text{-}int(A)$). Making use of δ^*-open sets, we introduce a new class of mappings called L (δ^*-open, open) mapping. This new class of mappings contains the class of strongly continuous mappings [**6**] properly and is contained in the classes of strongly θ-continuous mappings [**11**] as well as super continuous mappings [**10**].

3. Definitions and characterizations

3.1 Definition. A mapping $f : X \to Y$ is said to be **L (δ^*-open, open)** at $x \in X$, if to every open set M containing f (x), there exists a δ^*-open set N containing x such that $f(N) \subset M$. f is said to be L (δ^*-open, open) if it is L(δ^*-open, open) at each $x \in X$.

3.2 Definition. A set $G \subset X$, is said to be **δ^*-open neighborhood** [**14**] of $x \in X$, if there exists a δ^*-open set H such that $x \in H \subset G$.

3.3 Theorem. For a mapping $f : X \to Y$, the following are equivalent:
(a) f is L (δ^*-open, open).
(b) Inverse images of every open set is δ^*-open. i. e. f is I (δ^*-open, open).
(c) Inverse image of every closed set is δ^*-closed i.e. f is I (δ^*-closed, closed).
(d) $f(\delta^*\text{-}cl(A)) \subset cl(f(A))$ for each subset A of X.
(e) $\delta^*\text{-}cl(f^{-1}(B)) \subset f^{-1}(cl(B))$ for each subset B of Y.
(f) For each point x of X and for each neighborhood M of f(x), there exists a δ^*-open neighborhood N of x such that $f(N) \subset M$.
Proof. The proof is easy to establish.

3.4 Theorem. For a bijection $f : X \to Y$, the following are equivalents:
(a) F is L (δ^*-open, open)
(b) $Int(f(A)) \subset f(\delta^*\text{-}int(A))$ for each subset A of X.
(c) $f^{-1}(int(B)) \subset \delta^*\text{-}int(f^{-1}(B))$ for each subset B of Y.

Proof. (a) ⇒ (b). For each subset A of X, $int(f(A)) \subset f(A) \Rightarrow f^{-1}(int(f(A))) \subset A \Rightarrow \delta^*\text{-}int(f^{-1}(int(f(A)))) \subset \delta^*\text{-}int(A)$ or $f^{-1}(int(f(A))) \subset \delta^*\text{-}int(A)$. Thus, $int(f(A)) \subset f(\delta^*\text{-}int(A))$.

(b) ⇒ (c). For $B \subset Y$, $f^{-1}(B) \subset X$, so , $int(f(f^{-1}(B))) \subset f(\delta^*\text{-}int(f^{-1}(B)))$ or, $int(B) \subset f(\delta^*\text{-}int(f^{-1}(B))$ or, $f^{-1}(int(B)) \subset \delta^*\text{-}int(f^{-1}(B))$.

(c) ⇒ (a). If A is open in Y, then, $f^{-1}(int(A)) \subset \delta^*\text{-}int(f^{-1}(A)) \Rightarrow A \subset f(\delta^*\text{-}int(f^{-1}(A)) \Rightarrow f^{-1}(A) \subset \delta^*\text{-}int(f^{-1}(A))$. Thus $f^{-1}(A)$ is δ*-open.

Singal and Yadav [**14**] have shown that the collection of all clopen sets in a space (X, ℑ) forms a base for a topology ℑ* and A is ℑ*-open iff A is δ^*-open in (X, ℑ). The subtopology ℑ* is called **O-dimensionalization** [**14**] of ℑ. Also, a space is O-dimensional iff ℑ = ℑ*. Using this, we have

3.5 Theorem. For a mapping f : (X, ℑ) → Y, the following are equivalent:

(a) f is L (δ^*-open, open).

(b) f : (X, ℑ*) → Y is continuous.

(c) f is continuous provided X is O-dimensional.

Proof. Obvious.

4. Comparisions

4.1 Definition. A mapping f : X → Y is said to be **super continuous** [**10**] (resp. **Completely continuous** [**1**], **totally continuous** [**4**], **strongly θ-continuous** [**11**]) if inverse image of every open set in Y is δ-open (resp. regular open, clopen, θ-open) set in X,

Since every δ^*-open set is δ-open, so every L (δ^*-open, open) mapping is super continuous but the converse is not true as shown in the following example:

4.2 Example. If X = {a, b, c}, ℑ = {ϕ, X, {a}, {b}, {a, b}}.Then the identity map on (X, ℑ) is super continuous but not L (δ^*-open, open).

Singal and Yadav [**14**] have shown that every regular open set is clopen in an extremally disconnected space. So we conclude that every δ-open set is δ^*-open in an extremally disconnected space and hence every super continuous (or completely continuous) mapping on an extremally disconnected space is L (δ^*-open, open).

The following example shows the necessity of extremally disconnectedness of the domain space.

4.3 Example. Let X = {p, q, r, s} and ℑ = {ϕ, X, {p}, {q, r}, {p, q, r}}, and Y= {a, b, c} with ψ = {ϕ, Y, {a}, {c}, {a, c}}. Then f : (X, ℑ) → (Y, ψ) defined by f(p) = b, f(q) = a = f (r), f(s) = b is completely continuous and hence super continuous but not L (δ^*-open, open).

Moreover, every δ^*-open set is θ-open, therefore, every L (δ^*-open, open) mapping is strongly θ-continuous but the converse is not true in general, as following example shows.

4.4 Example. The identity map on the real line R is strongly θ-continuous but not L (δ^*-open, open).

Since closure of an open set is open in an extemally disconnected space, so every θ-open set is δ^*-open and hence every strongly θ-continuous mapping on an extremally disconnected space is L (δ^*-open, open).

4.5 Definiton. A mapping f : X → Y is said to be **strongly continuous** [**6**] (resp. **strongly semicontinuous** [**16**]) if $f(cl(A)) \subset f(A)$ (resp. $f(s\text{-}cl(A)) \subset f(A)$) for every subset A of X.

Obviously, every strongly continuous map is L (δ^*-open, open) but the converse is true provided the domain space is discrete, otherwise, the following example.

4.6 Example. If X = {a, b} with indiscrete topology on X, then the identity map on X is L (δ^*-open, open) but not strongly continuous.
The mappings being L (δ^*-open, open) and being strongly semicontinuous are independent concepts as the following example shows:

4.7 Example [**16**]. Let X be the set of reals with the usual topology and let Y = {a, b} with discrete topology. The map f : X → Y defined by

$$f(x) = \begin{cases} a \text{ if } -\infty < x < 0 \\ b \text{ if } 0 < x < \infty \end{cases}$$

is strongly semicntinuous but not L (δ^*-open, open). The L (δ^*-open, open) mapping discussed in **Example 4.6** is not even strongly semi-continuous.

The following example shows that even a homeomorphism may fail to be L (δ^*-open, open).

4.8 Example. Let X = {x, y, z}, $\mathfrak{I}$ = {ϕ, X, {y}, {z},{y, z}}. Then the identity map on (X, $\mathfrak{I}$) is homeomorphism but not L (δ^*-open, open). It also shows that a bijective na-continuous map carrying δ-open (resp. δ-closed) sets on to δ-open (resp. δ-closed) sets may fail to be L (δ^*-open, open).

Thus we have the following Implication diagram:

strongly continuous mappings
↓
strongly semcontinuous mappings ↮ L (δ^*-open, open) mappings → super continuous mappings

↓

strongly θ-continuous mappings

↓

continuous mappings

Where, → indicates implies but not implied by and ↮ indicates independence.

5. Properties of L (δ^*-open, open) mappings

5.1 Theorem.

(a) Every constant map is L (δ^*-open, open).

(b) Every mapping with domain a discrete space is L (δ^*-open, open).

(c) Every mapping with codomain an indiscrete space is L (δ^*-open, open).

(d) If $f : X \to Y$ is L (δ^*-open, open) then it remains so if the topology on X is replaced by a finer topology and / or the topology on Y is replaced by a coarser topology.

5.2 Definition. A mapping is said to be **S-continuous** [**15**] (resp. **semi continuous** [**7**], **slightly semicontinuous** [**14**], **feeble continuous** [**3**], **almost feeble continuous** [**12**], **faintly continuous** [**8**]) iff inverse image of every semiopen (resp. open, clopen, open, δ-open, θ-open) set is open (resp. semiopen, semiopen , feebly open, α-open (or feebly open) open) set respectively.

5.3 Definition. A mapping is said to be **D (δ^*-open, δ-open)** (resp. **D (δ^*-closed, δ-closed)**) if it maps δ^*-open (resp. δ^*-closed) sets on to δ-open (resp. δ-closed) sets.

5.4 Theorem. Regarding composite mappings, the following are easy to establish.

For $f : X \to Y$, and $g : Y \to Z$, gof is

(a) L (δ^*-open, open) whenever f is L (δ^*-open, open) and g is continuous.

(b) Semicontinuous whenever f is slightly semicontinuous and g is L (δ^*-open , open).

5.5 Theorem.

(a) If $g : Y \to Z$ is an open map, $gof : X \to Z$ is L (δ^*-open, open) then $f : X \to Y$ is L (δ^*-open , open).

(b) If $f : X \to Y$ is L (δ^*-open , open) and D (δ^*-open , open) map and $g : Y \to Z$ then gof is L (δ^*-open , open) iff g is L (δ-open ,open).

(c) Let $f : X \to Y$ be L (δ^*-open, open) surjection and $g : Y \to Z$. If gof is D (δ^*-open, δ -open) , (or D (δ^*-closed , δ-closed) so is g.

(d) Let $f: X \to Y$ and $g: Y\to Z$ be L (δ^*-open, open) injection. If gof is D (δ^*-open, δ-open (or D (δ^*-closed, δ-closed) so is f.

5.6 Corollary.
(a) Composition of two L (δ^*-open, open) mappings is L (δ^*-open, open).
(b) If $f : X \rightarrow Y$ is faintly continuous, $g : Y \rightarrow Z$ is L(δ^*-open, open) then gof is continuous.
(c) If $f : X \rightarrow Y$ is almost feeble continuous and $g : Y \rightarrow Z$ is L(δ^*-open, open), then gof is feeble continuous.

5.7 Theorem. Restriction of an L (δ^*-open, open) map is L (δ^*-open, open).
Proof. Let $f : X \rightarrow Y$ be L (δ*-open, open). $A \subset X$, $f_A : A \rightarrow Y$, Let G be open in Y, then f^{-1} (G) is δ^*-open in X . To show $f_A^{-1}(G) = f^{-1}(G) \cap A$, δ^*-open in A. If $x \in f_A^{-1}(G)$, then there exists a clopen set H such that $x \in H \subset f^{-1}(G)$ and hence $x \in (H \cap A) \subset f_A^{-1}(G)$ showing $f_A^{-1}(G)$ is δ^*-open in A.

5.8 Theorem. Let $X = A \cup B$, where A and B are clopen sets in X. Let $f : A \rightarrow Y$ and $g : B \rightarrow Y$ be L (δ^*-open, open). If f (x) = g(x) for every x in $A \cap B$, then $h : X \rightarrow Y$ defined by h (x) = f(x) if x in A and h(x) = g(x) if x is in B, is L (δ^*-open, open).
Proof. Let G be open in Y. Then $h^{-1}(G) = f^{-1}(G) \cup g^{-1}(G)$ is δ^*-open in X as both $f^{-1}(G)$ and $g^{-1}(G)$ are δ^*–open in clopen sets A and B respcdivaly and hence δ*-open in X.

5.9 Corollary.

(a) Let $X = \cup \{A_\lambda : \lambda \in \wedge\}$, where $A_\lambda{}^s$ are clopen and pairwise disjoint and $f_\lambda : A_\lambda \rightarrow Y$ be L (δ^*-open, open) for each λ. Then $h : X \rightarrow Y$ defined by $h(x) = f_\lambda(x)$ if $x \in A_\lambda$ is L(δ^*-open, open).
(b) Let $X = \cup X_i$ where $X_i{}^s$ are clopen sets in X. Then $f: X \rightarrow Y$ is L (δ^*-open, open) is L (δ^*-open, open) iff the restriction f_{X_i} is L (δ^*-open, open) for each i.

5.10 Theorem. If the graph map $g : X \rightarrow X \times Y$ of $f : X \rightarrow Y$ defined by g(x) = (x, f(x)) for each $x \in X$ is L (δ^*-open, open) then so is f.
Proof. If $x \in X$ and V is any open set containing f (x) then $X \times V$ is open in $X \times Y$ containing (x, f(x)), so, there exists a δ^*-open set U in X, such that $g(U) \subset X \times V$. hence f $(U) \subset V$.

5.11 Theorem. Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be L (δ^*-open, open) maps. If $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$ then $f : X \rightarrow Y$ defined by $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ is L (δ^*-open, open).
Proof. Let G_1 be open in Y_1 and G_2 be open in Y_2. Then $G_1 \times G_2$ is basic open in Y and hence $f_1^{-1}(G_1) \times f_2^{-1}(G_2)$ is δ^*–open in X.

5.12 Corollary. Let $\{X_\lambda : \lambda \in \wedge\}$ and $\{Y_\lambda : \lambda \in \wedge\}$ be two families of topological spaces. If $f_\lambda : X_\lambda \to Y_\lambda$ is L (δ^*-open, open) then the map $f : \prod X_\lambda \to \prod Y_\lambda$ defined by $f(\{x_\lambda\}) = \{f_\lambda(x_\lambda)\}$ is L (δ^*-open, open) for each λ.

5.13 Theorem. If $f : X \to \prod X_\lambda$ is L (δ^*-open, open) then $P_\lambda of : X \to X_\lambda$ is L (δ^*-open, open) and conversely where P_λ is λth projection.
Proof. Since all projections are continuous, so the composition map $P_\lambda of$ is L (δ^*-open, open). Conversely, if $G = p_\lambda^{-1}(U)$ for some λ is any member of the defining subbase for the product space, then $f^{-1}(G) = f^{-1}(p_\lambda^{-1}(U)) = (p_\lambda of)^{-1}(U)$ is δ^*-open in X.

5.14 Theorem. A map $f : X \to (Y, U)$ is L (δ^*-open, open) iff $f : X \to (f(X), U_{(f(X)})$ is L (δ^*-open , open) where $U_{f(X)}$ is relativization of U to f(X).
Proof. If H is open in $U_{f(X)}$ then $H = G \cap f(X)$ for some open set G in (Y, U) and $f^{-1}(H) = f^{-1}(G)$ is δ^*-open in X. The converse is obvious.

5.15 Theorem. Let $f : X \to Y$ be L (δ^*-open, open). If Z is a space having Y as a subspace then the map $h : X \to Z$ is L (δ^*-open, open).
Proof. Let S be open in Z and $g : Y \to Z$ be the inclusion map. Then $g^{-1}(S)$ is open in Y and $f^{-1}(g^{-1}(S)) = (gof)^{-1} S = h^{-1}(S)$ is δ^*-open.

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