Some Simple Measures of Information And New Bi-measures of Entropy and Its Directed Divergence

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Abstract: Our endeavor here is to obtain some simple non parametric additive and non-additive measures of entropy, simple non-parametric measures of directed divergence. A bi-measure of Entropy is the sum of two function, containing one of which being a measure of entropy, is a measure of entropy and some new three parametric measure of entropy ;and thus ,directed divergence are derived.

Index Terms -Entropy, Directed Divergence, Additive, Inaccuracy, Bi-measures, weighted information improvement, concave, AMS subject classification-

1. INTRODUCTION

In 1948, Claude E. Shannon [1], the father of Information theory, came up with his transformative paper entitled "The Mathematical Theory of Communication" in *Bell System Technical Journal* that justified itself as a source to the twin discipline of information theory and coding theory and a decisive influence in the new branch of 'Information Theory'. The conception of information theory application in communication theory was put forward by Shannon [1] in 1940 its function in the area of digitization foundation of communication and information theory. This theory of Shannon [1] readily received by a number of researchers. In 1949 with C. Shannon [1] gave their innovative and crucial contribution in the information theory. Shannon's [1] measure of uncertainty has remarkable executions in the diverse domains. A very vital application of it is to maximize the number of messages, containing effective codes, over a noiseless channel time efficiently. A lot of information theoretical and applied statistical inference and data processing problems contain vital influence of the information measures. Obviously, the literature on enhancement of applications measures has broaden significantly of late.
The objective of the present paper is to explore the measure of entropy, measures of directed divergence.
The section two discusses the properties of basic concepts and properties of measures of entropy and measures of directed divergence; the section introduces new simple measures of entropy and examine its properties which is defined in section 2.1, with exploring their concavity property. Here, we expand the result of measures of directed divergence; examine the properties of directed divergence and derive the measure of directed divergence corresponding to first measure of entropy. The section 4 introduces new Bi-measures of entropy, section 4.1 brings up new three parametric bi-measures of entropy and examines their concavity property with attainment of five different limiting cases of parameters a, b, and k. Here we derive the directed divergence corresponding to three parametric bi-measure of entropy. The section 4.3 derives the generalized three parametric bi-measure of directed divergence; examines their concavity property and also obtains the limiting cases of parameters a, b, and k by introducing new parameter A. The section 5 concludes with vital remarks and applications of obtained measures of entropy and directed divergence. The section 6 cites the various references.

RESEARCH METHODOLOGY

A brief overview, investigations and applications of the information theory and communication theory constitute our proposed methodology for this project. With the application of empirical method, we have done a minute inspections of the exemplary research papers published in various journals and proceedings. Through observation and collection of data for communication process, we have applied some parametric and general cases of entropy and directed divergence. Ensuring justified synthesis of objective and descriptive method, problem posing and problem solving method are practiced. Some parametric and unique generalization of communication theories are also carried out.

2. PRELIMINARIE

2.1 Measure of Entropy

Let $P = (p_1,p_2,\ldots\ldots,p_n)$, $\sum_{i=1}^{n} p_i = 1$, $p_i \geq 0$, i = 1,----,n (2.1.1)
be a probability distribution then, the corresponding measure of entropy given by Shannon [1] as,
$H_1(P) = -\sum_{i=1}^{n} p_i ln p_i$ (2.1.2)
The above expression is called as a measure of entropy for measuring the uncertainty or entropy of the probability distribution. For any probability distribution, $P = (p_1,p_2,\ldots\ldots,p_n)$ a measure of entropy needs to satisfy for the following conditions.

i) It should be a continuous function of $p_1,p_2,\ldots\ldots,p_n$ since if $p_1,p_2,\ldots\ldots,p_n$ change by small amounts then their maximum value changes by a small amounts.
ii) It should be permutationally symmetric function of $p_1,p_2,\ldots\ldots,p_n$ since if $p_1,p_2,\ldots\ldots,p_n$ are permuted among themselves, thereupon their maximum value does not change.
iii) Since maximizing suggests minimizing the value $p_{max,}$ it should be maximum subject to $\sum_{i=1}^{n} p_i = 1$ when $p_1 = p_2 = \ldots = p_n = 1/n$ and it maximum value increases with n.

iv) It should be always non negative and vanishes when, $p_i = q_i$, for all i = 1,….,n and its maximum value should be zero for each n degenerate distribution.

Let P = (p_1,p_2,…….,p_n), $\sum_{i=1}^{n} p_i = 1$, $p_i \geq 0$, i = 1,----,n
Renyi [2] defined the parametric measure of entropy as

$$H_2(p) = \frac{1}{1-\alpha} ln \sum_{i=1}^{n} p_i^{\alpha} \quad (2.1.3)$$

Later, Havarda Charvat's [3] defined parametric measures of entropy as

$$H_3(p) = \frac{1}{1-\alpha} (\sum_{i=1}^{n} p_i^{\alpha} - 1) \quad (2.1.4)$$

Kapur [4] defined following parametric measures of entropy,

$$H_4(p) = -\sum_{i=1}^{n} p_i ln p_i + \frac{1}{a}[\sum_{i=1}^{n}(1 + ap_i) ln(1 + ap_i) - (1 + a)\ln(1 + a)] \quad (2.1.5)$$

Kapur [5]defined the following parametric measure of entropy this measure of entropy is the second generalization of Shannon [1],viz

$$H_5(p) = -\sum_{i=1}^{n} p_i ln p_i + \frac{1}{a^2}[\sum_{i=1}^{n}(1 + ap_i) ln(1 + ap_i) - (1 + a) ln(1 + a)] \quad (2.1.6)$$

It is easily verified that
if a→0 in (1.1.5) and (1.1.6) then
$H_4(P) \rightarrow H_1(P)$
and $H_5(P) \rightarrow H_1(P)$
Here $H_4(P)$ and $H_5(P)$ are the one parametric bi-measures of entropy, Since these are the sum of two functions in which first function is a measures of entropy, $H_1(P)$.

2.2 Measure Of Directed Divergence

Here, we discuss about some parametric and non-parametric measures of directed divergence and its properties, if P = (p_1,p_2-----,p_n) and Q = (q_1,q_2------,q_n) are two probability distributions, then the measures of directed divergence of P from Q is a function of D(P; Q) satisfy the following conditions:

a) D(P :Q) ≥ 0
b) D(P :Q) = 0 if and only if P = Q
c) D(P:U) = max H(P) – H(P),
Where U = (1/n , ----- ,1/n) is the uniform distribution.

If $-\sum_{i=1}^{n} \emptyset(p_i)$ is a parametric measure of entropy,
Where
i. ϕ (x) is a twice differentiable convex of x with
ii. ϕ(1) = 0
iii. ϕ(0) =0 ,

In the field of science and engineering, the "concept of distance" is considered very crucial; and it is extended for the concerns in the disciplines of economic, sociology, psychology, linguistics, genetics, biology, etc. However, here the motioned distance is not necessarily a geometrical distance but something of relative and probable significance. Hence, we need a modification when we have to consider a measure for the concept of distance between two probability distributions. A measure D(P:Q) of divergence or directed divergence is defined as the discrepancy of the probability distribution P from another probability distribution Q. Similarly, it measures the distance of P from Q.

A measure is a convex function of both P and Q is obtained from Csiszer's [6] measure

$$D_1(P: Q) = \sum_{i=1}^{n} q_i \phi(\frac{p_i}{q_i})$$

The most important and useful measure of directed divergence is given by the well-known Kullback-Leibler [7] derived measure of directed divergence $D_1(P: Q)$ of P from Q, corresponding to Shannon [1] measure of entropy as

$$D_1(P: Q) = \sum_{i=1}^{n} p_i ln \frac{p_i}{q_i} \quad (2.2.1)$$

Some parametric measures of directed divergence are:

$$D_{\alpha}(P: Q) = \frac{1}{\alpha-1} \ln \sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}, \alpha \neq 1, \alpha > 0 \quad (2.2.2)$$

It is Renyi's [2] probabilistic measure of directed divergence.

$$D^{\alpha}(P: Q) = \frac{1}{\alpha-1}[\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} - 1], \alpha \neq 1, \alpha > 0 \quad (2.2.3)$$

It is Havrda and Charvat's [3] probabilistic measure of divergence

Corresponding to $H_4(P)$ and $H_5(P)$ Kapur [4, 5] defined measures of directed divergence as

$D_4(P:Q) = \sum_{i=1}^{n} p_i ln\frac{p_i}{q_i} - \frac{1}{a}\sum_{i=1}^{n}(1+ap_i)ln\left(\frac{1+ap_i}{1+aq_i}\right)$, a >0 (2.2.4)

And

$D_5(P:Q) = \sum_{i=1}^{n} p_i ln\frac{p_i}{q_i} - \frac{1}{a^2}\sum_{i=1}^{n}(1+ap_i)ln\left(\frac{1+ap_i}{1+aq_i}\right)$, a > 0 (2.2.5)

Here D_4 (P: Q) and D_5 (P: Q) are one parametric measures of directed divergence. In this paper, we have extended $H_4(P)$ and $H_5(P)$ to get three parametric bi-measures of entropy including $H_4(P)$ and $H_5(P)$ as special cases. We have also derived three parametric bi-measures of directed divergence including D_4 (P: Q) and D_5(P: Q) as special cases. The object of the given paper is to investigate the parametric and non-parametric measure of entropy and its directed divergence, it is the comparative study of Kapur [4, 5, 8] measure of entropy by initiating new parameters in simple way.

3. SIMPLE MEASURE OF ENTROPY

3.1 First Simple Measure Of Entropy

Let, P = {$p_1,p_2,\ldots\ldots,p_n$ } be a probability distribution thereafter the measure of entropy for distribution are defined as as

$H_6(P) = -ln\frac{1}{\max_{1\le i\le n} p_i}$ (3.1.1)

The above measure (3.1.1) satisfies the following condition

i) $H_6(P)$ is defined for all p_i , $1 \le i \le n$,
ii) $H_6(P)$ is continuous for all p_i , $1\le p_i \le n$,
iii) It is permutationally symmetric function of $p_1, p_2,\ldots\ldots,p_n$.
iv) It is maximum subject to $\sum_{i=1}^{n} p_i = 1$
v) It is always non negative and its maximum value occurs zero is for the n degenerate distribution.
vi) To examine its concavity property, consider the measure of entropy (3.1.1) as,

$\emptyset(x) = -ln.\frac{1}{x}$

$\emptyset'(x) = \frac{1}{x}$

$\emptyset''(x) = -\frac{1}{x^2} \le 0$

$\therefore \emptyset''(x) \le 0$
$\therefore \emptyset''(x) \le 0$ (3.1.2)

So that $H_6(P)$ is a concave function of $p_1,p_2,\ldots\ldots,p_n$ ..

3.2 Second Simple Measure Of Entropy

Similarly, we have defined another measure entropy as

$H_7(P) = 1 - \max_{1\le i\le n} p_i$ (3.2.1)

The above measure (3.2.1) satisfies following condition

i) $H_7(P)$ is defined for all p_i , $1 \le i \le n$
ii) The above measure $H_2(P)$is a continuous function of $p_1,p_2,\ldots\ldots,p_n$ since if $p_1,p_2,\ldots\ldots,p_n$ changes then their maximum value changes.
iii) It is permutationally symmetric function of $p_1,p_2,\ldots\ldots,p_n$ since if $p_1,p_2,\ldots\ldots,p_n$ are permuted among themselves, then their maximum value does not change
iv) It is maximum subject to $\sum_{i=1}^{n} p_i = 1$ when $p_1= p_2 =\ldots.= p_n = 1/n$ and its maximum value is increases with n.
v) It is always non negative and vanishes when, $p_i = q_i$ for all i = 1,….,n and its minimum value should occur zero for each n degenerate distribution.

To examine its concavity property, consider the measure of entropy (3.2.1) as,

$\emptyset(x) = 1 - x$ (3.2.2)
$\emptyset''(x)=0$ (3.2.3)

Hence $H_7(P)$ is a concave function $p_1,p_2,\ldots\ldots,p_n$.
Here $\emptyset''(x) = 0$ therefore $H_7(P)$ is a concave function of $p_1,p_2,\ldots\ldots,p_n$.

3.3 Frist Measure Of Directed Divergence

Let $P = (p_1, p_2, \dots . p_n), \sum_{i=1}^{n} p_i = 1, \quad p_i \geq 0,$ i = 1,….,n be a probability distribution and Q = $(q_1, q_2, \dots . q_n)$be another probability distribution, then the measure of directed divergence of P from Q is D_6(P; Q) and D_7(P; Q).
D_6(P; Q) is a directed divergence corresponding to the first measure of entropy D_6(P:Q) satisfying the properties discussed in section three
From (3.1.2)
$\emptyset(x)$is twice differentiable convex function

$\emptyset(x) = -ln.\frac{1}{x},$
$\emptyset(1) = 0$
$\emptyset(0) = 1$ (3.3.1)

D_6(P :Q) ≥ 0
D_6(P :Q) = 0 if and only if P = Q
D_6(P:Q) is a convex function of P and Q

$\therefore D_6 (P:Q) = q_i ln \max_{1\leq i\leq n} \left(\frac{q_i}{p_i}\right)$ (3.3.2)

Hence, it can easily be verified that it has satisfied all the properties of measure of directed divergence.

3.4 Second Measure Of Directed Divergence

D_7(P; Q) is the directed divergence corresponding to the second measure of entropy obtained in this section satisfying the following conditions:

D_7(P:Q) satisfied properties discussed in section two
From (3.2.3)
$\emptyset(x)$is twice differentiable convex function

$\emptyset(x) = 1 - x \, , \emptyset(1) = 0 \, , \emptyset(0) = 1$ (3.4.1)
D_7(P :Q) ≥ 0
D_7(P :Q) = 0 if and only if P = Q
D_7(P:Q) is a convex function of P and Q

$D_7(P:Q) = q_i \left(1 - \max_{1\leq i\leq n} \frac{P_i}{q_i}\right)$ (3.4.2)
Hence it can easily be verified that D_7(P:Q) has satisfied all the properties of measure of directed divergence

4. Bi-Measures of Entropy

4.1 New Three Parametric Bi-Measure Of Entropy

Consider the new three parametric bi-measure of entropy as :
$H_{a,b,k}(P) = -\sum_{i=1}^{n} p_i ln p_i + \frac{b}{a^k}\left[\sum_{i=1}^{n}(1 + ap_i)ln(1 + ap_i) - p_i(1 + a)ln(1 + a)\right]$
Where a,b,k are parameters , a > 0, 0 < b ≤ 1 (4.1.1)
Now (4.1.1) can be written as $\sum_{i=1}^{n} f(p_i)$

$f(p_i) = - p_i ln p_i + \frac{b}{a^k}[(1 + ap_i)ln(1 + ap_i) - p_i(1 + a)ln(1 + a)]$ (4.1.2)

$f'(p_i) = - 1 - ln p_i + \frac{ab}{a^k}[1 + ln(1 + ap_i)] - (1 + a)ln(1 + a)$

$f''(p_i) = -\frac{1}{p_i} + \frac{a^2 b}{a^k}\left[\frac{1}{(1+ap_i)}\right]$ (4.1.3)
OR
$f''(p_i) = -\frac{(1+(a-C)p_i)}{p_i(1+ap_i)}$, where $C = \frac{b}{a^{k-2}}$ (4.1.4)

Hence $H_{a,b,k}(p)$ will be concave ,
if 1+ (a –C) $p_i \geq 0$ (4.1.5)
But this is true when
1 ≥C or $a \geq C$

i.e. When $a^{k-2} \geq$ b or $a^{k-1} \geq$b (4.1.6)

Some Special Cases

Case I

When

$a^{k-2} \geq b or a^{k-1} \geq b,$

$H_{a,b,k}$ (P) represents a three parametric bi-measure of entropy.

Case II

When b = 1, k=1,

$H_{a,b,k}(P) = -\sum_{i=1}^{n} p_i ln p_i + \frac{1}{a}[\sum_{i=1}^{n}(1 + ap_i)ln(1 + ap_i) - p_i(1 + a)ln(1 + a)]$

Hence, $H_{a,b,k}$ (P) = H_4(P) at b=1,k=1

Case III

When b =1, k = 2

$H_{a,b,k}(P) = -\sum_{i=1}^{n} p_i ln p_i + \frac{1}{a^2}[\sum_{i=1}^{n}(1 + ap_i)ln(1 + ap_i) - p_i(1 + a)ln(1 + a)]$

Hence, Ha,b,k(P) = H_5(P) , at b=1 and k =2

Case IV

When 0 < b ≤ 1 then (4.1.5) will hold if

(k-2) lna ≥ 0 or (k – 1) $ln\, a$ ≥ 0 (4.1.7)

But (k-2) a ≥ 0 ,if a≥ 1, k ≥ 2 or 0 <a <1, k < 2 ,

And (k – 1) ≥ 0 , if a ≥ 1, k≥ 1 or 0<a<1 , k<1

Case V

When b > 1, we can find suitable values of a, k using (4.1.7)

Thus, in any case mentioned above the family of bi-measures of entropies is much larger than H_4(P) and H_5(P).

4.2 New Bi-measures of Directed divergence

Let$P_i = (p_1, p_2, \ldots . p_n) \sum_{i=1}^{n} p_i = 1, \quad p_i \geq 0$ i = 1,….,n be a probability distribution and

Q = $(q_1, q_2, \ldots . q_n)$ be another probability distribution then the measure of directed divergence of P from Q is $D_{a,b,k}$(P:Q).

$D_{a,b,k}$(P:Q) is a directed divergence corresponding to the bi-measure of entropy $H_{a,b,k}$(P) obtained in section two

$D_{a,b,k}$(P:Q) satisfying the properties discussed in section one as

Now $H_{a,b,k}$(P)can be expressed as $\emptyset(x)$

where$\emptyset(x) = xlnx - \frac{b}{a^k}(1 + ax)ln\left(\frac{1+ax}{1+a}\right)$, a > 0 (4.2.1)

Then by (4.1.3),(4.1.4),(4.1.5) and (4.1.6) we find that $\emptyset(x)$ is a convex function if either of the condition (4.1.6) are satisfied.

$\emptyset(1) = ln1 - \frac{b}{a^k}(1 + a)ln\left(\frac{1+a}{1+a}\right)$

$\emptyset(1) = 0$

$\emptyset(0) = 0$

$D_{a,b,k}$(P:Q) ≥ 0

$D_{a,b,k}$(P:Q) = 0 if and only if P = Q

$D_{a,b,k}$(P:Q) is a convex function of P and Q

then by Csiszer's [6], the directed divergence for three parametric measure of entropy (4.2.2)

gives the family of three parametric bi-measures of directed divergence

D (P:Q) = $q_i \emptyset\left(\frac{p_i}{q_i}\right)$

$D_{a,b,k}$(P:Q)= $q_i\left[\sum_{i=1}^{n}\frac{p_i}{q_i}\, ln\frac{p_i}{q_i} - \frac{b}{a^k}\sum_{i=1}^{n}(1 + a\frac{p_i}{q_i})ln\frac{1+a\frac{p_i}{q_i}}{1+a}\right]$

$D_{a,b,k}$(P:Q)= $\sum_{i=1}^{n} p_i\, ln\frac{p_i}{q_i} - \frac{b}{a^k}\sum_{i=1}^{n}(q_i + ap_i)ln\left(\frac{q_i+ap_i}{q_i(1+a)}\right)$ (4.2.3)

$D_{a,b,k}$(P:Q) are different from D_4(P,Q) and D_5(P:Q) and these are superior to D_4(P:Q) and D_5(P:Q).thus, $D_{a,b,k}$(P:Q) are convex function of both P and Q and represent a much larger class of measures than D_4(P,Q) and D_5(P:Q).

4.3 Generlised Bi-Measure Of Directed Divergence

The Generlised bi-measure of directed divergence corresponding to bi-measure of entropy (4.1.1) by considering

$\emptyset(x) = xlnx + A(1 + ax)ln\left(\frac{1+ax}{1+a}\right)$, where A= $-\frac{b}{a^k}$ (4.3.1)

$\emptyset'^{(x)} = 1 + lnx + A\left(a + ln\left(\frac{1+ax}{1+a}\right).a\right)$ (4.3.2)

$\emptyset''^{(x)} = \frac{1}{x} + \frac{Aa^2}{1+ax}$ (4.3.3)

Will be convex if and only if

$\therefore a + Aa^2 > -\frac{1}{x}$ (4.3.4)

Now x can vary from 0 to ∞ , so that $\frac{-1}{x}$ can vary from $-\infty$ to 0,so that (4.3.4) becomes

$(a + Aa^2) > 0$

Or A> $-\frac{1}{a}$ (4.3.5)

Where A= $-\frac{b}{a^k}$

∴ (4.3.5) becomes

$-\frac{b}{a^k} > -\frac{1}{a}$

i.e b $< a^{k-1}$ (4.3.6)

∴ The generlised bi-measure of directed divergence function of P and Q. thus, $D_{a,b,k}(P:Q)$ is a convex function $D_{a,b,k}(P:Q) = \sum_{i=1}^{n} p_i \ln\frac{p_i}{q_i} - A\sum_{i=1}^{n}(q_i + ap_i)\ln\left(\frac{q_i+ap_i}{q_i(1+a)}\right)$

Where, A is any positive number or a negative number greater than or equal to $-\frac{1}{a}$

The generlised bi-measure of directed divergence $D_{a,b,k}$(P:Q) are different from D_4(P,Q) and D_5(P:Q) and these are superior to D_4(P:Q) and D_5(P:Q).thus, $D_{a,b,k}$(P:Q) are convex function of both P and Q and represent a much larger class of measures than D_4(P,Q) and D_5(P:Q).

4. CONCLUSION

- We can verify various properties of obtained measures of entropy and directed divergences also obtain measures of weighted directed divergence, measures of weighted information improvement, measures of inaccuracy of measure of entropy obtained in section two and three.
- In the present communication, the obtained measures are simplest possible which depend implicitly and explicitly on the maximum value of memberships or maximum values of ratio of memberships. Here, non-parametric measures of inaccuracy and non-parametric measures of information improvement, are also obtained
- The Bi-measures of entropy $H_{a,b,k}(P)$ introduced in section two satisfying all the properties discussed in section one
- On different values of a and b we get measure of entropy defined by Kapur [4], [5] which is H_4(p) and H_5(p) and in any case, the family of bi-measure of entropies is much larger than H_4(p) and H_5(p)
- This measure $H_{a,b,k}(P)$ gives the measures of directed divergence, the bi-measure of directed divergence $D_{a,b,k}$ (P:Q) satisfying all the properties discussed in section two. It is convex function of both P & Q and $D_{a,b,k}$ (P:Q) is different from D_4(P:Q) and D_5(P:Q). The directed divergence $D_{a,b,k}$(P:Q) can satisfy various other properties of directed divergence. The inaccuracy and weighted information improvement can also be found which represents a much larger class of measure than those of obtained directed divergence.
- The generalized bi-measure of directed divergence $D_{a,b,k}$(P:Q) meets all the properties discussed in section two. It is observed that it is convex function of both P & Q, and the generalized bi-measure of directed divergence $D_{a,b,k}$ (P:Q) is the generalized from D_4(P:Q) and D_5(P:Q). The generalized directed divergence $D_{a,b,k}$(P:Q) can satisfy various other properties of directed divergence.

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