CONVERGENCE AND STABILITY ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS VIA COLLOCATION TECHNIQUES

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ABSTRACT

The comprehensive numerical study has been made here for the solution of Partial differential Equations by using Hermite collocation method with Hermite basis function. In the numerical implementation, approximation solution and its space derivatives over the interval are approximated by the combination of the cubic Hermite interpolating polynomials and parameters. The accuracy, efficiently, simplicity, stability and reliability of HCM performs are demonstrated through some numerical examples. The results have been presented in tabular and graphical forms

1. Introduction

The problems associated with heat conductivity, mass transport, diffusion-dispersion is described with the assistance of ordinary or partial differential equations. These equations are treated as per static or dynamic nature. When the problem is defined for more than one point, it is termed as *boundary value problem*. In those situations where analytic solution does not exist or it is difficult to find the analytic solution, the numerical techniques are followed to obtain an approximate solution. A variety of numerical techniques have been followed by different investigators such as finite difference scheme (Gracia & Riordan 2012 and Luo *et al.* 2013), collocation method (Arora *et al.* 2005, Yang *et al.* 2014 and Fernandes *et al.* 2015), Galerkin method (Nadukandi *et al.* 2012 and Wei *et al.* 2012), and Finite element method (Parvazinia 2012) etc. to solve such type of initial and boundary value problems.

Among the above mentioned methods, orthogonal collocation method developed by Villadsen & Stewart (1967) is one of the simplest method and easily adaptable to computation procedure. It is one of the weighted residual methods which is used to discretize the initial and boundary value problems. In collocation technique, an unknown function $y(\xi,\tau)$ is approximated to satisfy the differential equation $\Im^V(y)=0$ along with the boundary conditions $\Im^B(y)=0$, where B is the boundary adjoining the volume V. The solution function y is approximated using a trial function $\overline{y}$, which is a linear combination of series of orthogonal polynomials. The residual function $\Re(\xi,\tau)$ is defined over the volume V along with the boundary B.

In the principle of collocation, to minimize the error, inner product of residual function to the weight function of the given base polynomial is set equal to zero at collocation points, *i.e.*, $\langle \Re(\xi,\tau), W(\xi) \rangle = 0$. It forces the residual to be equal to zero at the collocation points. Zeros of orthogonal polynomials are taken as collocation points. The dependent variables are the solution values at the collocation points instead of the coefficients in the expansion. The orthogonality property of the polynomials ensures that the zeros are real and distinct.

In the present study the technique of orthogonal collocation is followed in conjecture with the Hermite interpolating polynomials.

2. HERMITE COLLOCATION METHOD

In Hermite collocation method, the approximating function is discretized in terms of Hermite interpolating polynomials, as these polynomials possess continuity property at node points. Due to the continuity property of Hermite polynomials there is no need to assume that approximating function and its first derivative should be continuous at node points. Hermite collocation method has the property to transform the mixed collocation method into interior collocation method as the boundary conditions are satisfied at the initial and terminal point of the interval [0,1]. Hermite interpolating polynomials are defined for the tangent at node points, which reduces the error in interpolating the layered region.

In present study, cubic Hermite interpolating polynomials (Dyksen & Lynch 2000 and Brill 2002) have been followed to discretize the approximating function to interpolate the given initial and boundary value problems and are defined as:

$$P_j(\xi)=\begin{cases}3\left(\dfrac{\xi_{j+1}-\xi}{\xi_{j+1}-\xi_j}\right)^2-2\left(\dfrac{\xi_{j+1}-\xi}{\xi_{j+1}-\xi_j}\right)^3, & \xi_j\le\xi\le\xi_{j+1}\\ 3\left(\dfrac{\xi-\xi_{j-1}}{\xi_j-\xi_{j-1}}\right)^2-2\left(\dfrac{\xi-\xi_{j-1}}{\xi_j-\xi_{j-1}}\right)^3, & \xi_{j-1}\le\xi\le\xi_j\\ 0, & elsewhere\end{cases} \tag{1}$$

$$\overline{P}_j(\xi)=\begin{cases}\dfrac{\left(\xi_{j+1}-\xi\right)^2}{\xi_{j+1}-\xi_j}-\dfrac{\left(\xi_{j+1}-\xi\right)^3}{\left(\xi_{j+1}-\xi_j\right)^2}, & \xi_j\le\xi\le\xi_{j+1}\\ -\dfrac{\left(\xi-\xi_{j-1}\right)^2}{\xi_j-\xi_{j-1}}+\dfrac{\left(\xi-\xi_{j-1}\right)^3}{\left(\xi_j-\xi_{j-1}\right)^2}, & \xi_{j-1}\le\xi\le\xi_j\\ 0, & elsewhere\end{cases} \tag{2}$$

These piecewise cubics are designed such that $P_j(\xi_i)=\delta_{ji}$, $P_j{}'(\xi_i)=0$, $\overline{P}_j(\xi_i)=0$, $\overline{P}_j{}'(\xi_i)=\delta_{ji}$.

By rearranging the terms in eq. (1) and eq. (2), Hermite interpolating polynomials can be rewritten as:

$$H_i(\xi)=\begin{cases}3\left(\dfrac{\xi_{i+1}-\xi}{\xi_{i+1}-\xi_i}\right)^2-2\left(\dfrac{\xi_{i+1}-\xi}{\xi_{i+1}-\xi_i}\right)^3, & \xi_i\le\xi\le\xi_{i+1}\\ 3\left(\dfrac{\xi-\xi_{i-1}}{\xi_i-\xi_{i-1}}\right)^2-2\left(\dfrac{\xi-\xi_{i-1}}{\xi_i-\xi_{i-1}}\right)^3, & \xi_{i-1}\le\xi\le\xi_i\\ \dfrac{\left(\xi_{i+1}-\xi\right)^2}{\xi_{i+1}-\xi_i}-\dfrac{\left(\xi_{i+1}-\xi\right)^3}{\left(\xi_{i+1}-\xi_i\right)^2}, & \xi_i\le\xi\le\xi_{i+1}\\ -\dfrac{\left(\xi-\xi_{i-1}\right)^2}{\xi_i-\xi_{i-1}}+\dfrac{\left(\xi-\xi_{i-1}\right)^3}{\left(\xi_i-\xi_{i-1}\right)^2}, & \xi_{i-1}\le\xi\le\xi_i\\ 0, & elsewhere\end{cases} \tag{3}$$

The first order derivative of cubic Hermite basis is defined as:

$$H_i'(\xi)=\begin{cases} -6\dfrac{\xi_{i+1}-\xi}{\left(\xi_{i+1}-\xi_i\right)^2}+6\dfrac{\left(\xi_{i+1}-\xi\right)^2}{\left(\xi_{i+1}-\xi_i\right)^3}, & \xi_i\le\xi\le\xi_{i+1} \\ 6\dfrac{\xi-\xi_{i-1}}{\left(\xi_i-\xi_{i-1}\right)^2}-6\dfrac{\left(\xi-\xi_{i-1}\right)^2}{\left(\xi_i-\xi_{i-1}\right)^3}, & \xi_{i-1}\le\xi\le\xi_i \\ -2\dfrac{\xi_{i+1}-\xi}{\xi_{i+1}-\xi_i}+3\left(\dfrac{\xi_{i+1}-\xi}{\xi_{i+1}-\xi_i}\right)^2, & \xi_i\le\xi\le\xi_{i+1} \\ -2\dfrac{\xi-\xi_{i-1}}{\xi_i-\xi_{i-1}}+3\left(\dfrac{\xi-\xi_{i-1}}{\xi_i-\xi_{i-1}}\right)^2, & \xi_{i-1}\le\xi\le\xi_i \\ 0, & elsewhere \end{cases} \tag{4}$$

The behavior of Hermite cubics $P_0, P_1, \overline{P}_0, \overline{P}_1$ is shown in Figure 1.

3. COLLOCATION POINTS

In Hermite collocation, the choice of collocation points depends upon the degree of the interpolating polynomial as well. In present study, cubic Hermite polynomials have been taken as interpolating polynomials, therefore, two collocation points have been taken within each interval $[\xi_{j-1}, \xi_j]$ and $[\xi_j, \xi_{j+1}]$. To apply orthogonal collocation within each interval $[\xi_{j-1}, \xi_j]$ a new variable ζ is introduced in such a way that $\zeta=\dfrac{\xi-\xi_{j-1}}{h_j}$ where $h_j=\xi_j-\xi_{j-1}$ such that $\zeta=0$ when $\xi=\xi_{j-1}$ and $\zeta=1$ when $\xi=\xi_j$. Roots of shifted Legendre polynomial of order two have been calculated as:

$$\zeta_i=\frac{\sqrt{3}\pm1}{2\sqrt{3}}; \qquad i = 1, 2 \tag{5}$$

4. APPLICATION OF HERMITE COLLOCATION METHOD TO INITIAL BOUNDARY VALUE PROBLEMS

Consider the following initial boundary value problem:

$$y_\tau=\overline{\varepsilon}y_{\xi\xi}-\overline{\alpha}(\xi)y_\xi-\overline{\beta}(\xi)y+f(\xi) \qquad \forall\ (\xi,\tau)\in\Omega\times(0,\mathrm{T}) \tag{6}$$

where $y_\tau=\dfrac{\partial y}{\partial\tau}$, $y_{\xi\xi}=\dfrac{\partial^2 y}{\partial\xi^2}$, $y_\xi=\dfrac{\partial y}{\partial\xi}$, and $f(\xi)$ is a continuous function of ξ. $\overline{\alpha}(\xi)$ and $\overline{\beta}(\xi)$ are continuous functions of ξ such that $\overline{\alpha}(\xi)$ and $\overline{\beta}(\xi)$ are positive bounded functions possessing continuous derivatives for all $\xi\in\Omega$.

The boundary conditions along the layers are assumed to be of Robbinson's type or mixed conditions:

$$y-\overline{\varepsilon}y_\xi=0 \qquad \text{at } \xi=0,\ \forall\ \tau>0 \tag{7}$$

$$y_\xi=0 \qquad \text{at } \xi=1,\ \forall\ \tau>0 \tag{8}$$

Initially, it is assumed that $y_0 = y(\xi, 0) = 1;\ \forall\ \xi\in\Omega$ (9)

The cubic Hermite approximation is defined as:

$$y(\xi,\tau)=\sum_{i=1}^{m}a_i(\tau)H_i(\xi),\ \forall\ \xi\in[\xi_{i-1},\xi_{i+1}] \tag{10}$$

Where, $a_i(\tau)$'s are the continuous functions of 'τ'. The continuous functions $a_i(\tau)$'s are arranged in such a way in the interval $[\xi_{i-1}, \xi_i]$ and $[\xi_i, \xi_{i+1}]$ that the problem of double collocation omits out. The behaviour of these polynomials in the interval $[\xi_{i-1}, \xi_i]$ and $[\xi_i, \xi_{i+1}]$ is shown in Figure 2.

5. ERROR ANALYSIS

Discretization in 'τ' and 'ξ' direction

The error analysis in the present study is motivated by the work of (Swartz & Varga 1972 and Kumar & Kadalbajoo 2011). Consider the interval $[\tau_m, \tau_{m+1}]$ in 'τ' direction such that $\Delta\tau = |\tau_m - \tau_{m+1}|$. Discretizing the equation (6) in 'τ' direction using finite difference scheme such that equation (6) takes the form:

$$\frac{y_{m+1} - y_m}{\Delta\tau} = \bar{\varepsilon}\frac{d^2 y_{m+1}}{d\xi^2} - \bar{\alpha}(\xi)\frac{dy_{m+1}}{d\xi} - \bar{\beta}(\xi)y_{m+1} + f(\xi) \tag{11}$$

Equation (4.23) takes the following form

$$g(\xi, \tau_m) = \bar{\varepsilon}\frac{d^2 y_{m+1}}{d\xi^2} - \bar{\alpha}(\xi)\frac{dy_{m+1}}{d\xi} - \gamma y_{m+1} \tag{12}$$

where $g(\xi, \tau_m) = - f(\xi) - y_m/\Delta\tau$ and $\gamma = \bar{\beta}(\xi) + 1/\Delta\tau$ and $y_0 = y(\xi, 0) = 1$.

To apply the collocation technique, interval [0, 1] is divided into small subintervals $[\xi_{i-1}, \xi_i]$. Define $H(\xi)$ to be the space of all Hermite interpolating polynomials satisfying (6). Let $y(\xi)$ be the exact solution of the steady state part of the differential equation defined by eq. (6):

$$\bar{\varepsilon}\frac{d^2 y}{d\xi^2} - \bar{\alpha}(\xi)\frac{dy}{d\xi} - \gamma y = g(\xi) \tag{13}$$

Let $\bar{y}(\xi)$ be the collocation solution of (6). After applying the orthogonal collocation within each subdomain, the values of $\bar{y}(\xi)$, $\bar{y}'(\xi)$ and $\bar{y}''(\xi)$ are substituted into the eq. (6) to obtain a system of equations:

$$\boldsymbol{Ha = g} \tag{14}$$

Where $\boldsymbol{a}$ is the matrix of collocation constants a_i's, $\boldsymbol{H}$ is the matrix of collocation equations defined at collocation points and $\boldsymbol{g}$ is the matrix of function of ξ defined at collocation points such that:

$$H_{ji} = \left\{ \frac{\bar{\varepsilon}}{h^2} B_{ji} - \frac{\bar{\alpha}_j}{h} A_{ji} - \gamma_j ; \quad i, j = 1, 2, \ldots m \right. \tag{15}$$

where $\bar{\alpha}_j = \bar{\alpha}(\xi_j)$ and $\gamma_j = \gamma(x_j)$, B_{ji} and A_{ji} are the discretized matrices of second and first order derivatives of $\bar{y}(\xi)$ at collocation points. The boundary conditions involve the gradient at boundary points and are therefore satisfied by the approximating function $\bar{y}(\xi)$ using the properties of Hermite interpolating polynomials defined in eq. (1) and eq. (2). This is the property of Hermite collocation method which differentiates it from orthogonal collocation method and orthogonal collocation on finite elements using Lagrange's basis.

The system of equations defined by eq. (14) gives a block diagonal structure of matrix $\boldsymbol{H}$ of order $2m\times 2m$. The bandwidth of each block is 2. The matrix structure of matrix $\boldsymbol{H}$ is defined in Figure 3. The matrix $\boldsymbol{H}$ being diagonally dominant is non-singular. Hence the method of collocation using cubic Hermite basis has a unique solution of system (14) defined by $\bar{y}(\xi)$.

6. RATE OF CONVERGENCE

Next step of the present study is the determination of rate of convergence of the given numerical technique of Hermite collocation. To determine the rate of convergence, method of (Farrell & Hagarty 1991) has been followed. Define the maximum pointwise error as:

$$E_{\bar{\varepsilon}}^{m} = \left\| \bar{y}^{m} - \bar{y}^{2m} \right\|_{\infty} \tag{16}$$

where $\bar{y}^{m}$ is the cubic Hermite approximation of $y(\xi, \tau_k)$ at the m node points. The additional node points in the spatial direction can be added by selecting the mid points of the node points ξ_i's in the spatial direction for $1 \le i \le m$. The $\bar{\varepsilon}$-uniform error is defined as:

$$E^{m} = \max_{\bar{\varepsilon}} E_{\bar{\varepsilon}}^{m} \tag{17}$$

The rate of convergence is calculated as:

$$p_{\bar{\varepsilon}}^{m} = \frac{\log(E_{\bar{\varepsilon}}^{m}) - \log(E_{\bar{\varepsilon}}^{2m})}{\log(2)} \tag{18}$$

The $\bar{\varepsilon}$-uniform rate of convergence is calculated as:

$$p^{m} = \frac{\log(E^{m}) - \log(E^{2m})}{\log(2)} \tag{19}$$

7. NUMERICAL EXAMPLES

Problem 7.1

Consider linear advection-diffusion equation with mixed boundary conditions and

$\bar{\alpha}(\xi) = \bar{\beta}(\xi) = 1,\; f(\xi) = \cos \pi\xi.$

$$\frac{\partial y}{\partial \tau} = \bar{\varepsilon} \frac{\partial^2 y}{\partial \xi^2} - \frac{\partial y}{\partial \xi} - y + \cos \pi\xi \tag{20}$$

Boundary conditions are given as:

$$y(0,\tau) - \bar{\varepsilon} \left. \frac{\partial y}{\partial \xi} \right|_{\xi=0} = 0, \qquad \forall\ \tau \in (0,T] \tag{21}$$

$$\left. \frac{\partial y}{\partial \xi} \right|_{\xi=1} = 0, \qquad \forall\ \tau \in (0,T] \tag{22}$$

Initially, it is assumed that $y(\xi, 0) = 1$, $\forall\ \xi \in \Omega$. In Figure 4, the behaviour of solution profiles is shown at different time intervals for $\bar{\varepsilon} = 2^{-4}$. In Figure 5, the behaviour of solution profiles is shown for different values of $\bar{\varepsilon}$ for 32 mesh points at $\xi = 1$. The solution profile for $\bar{\varepsilon} = 0.1$ converge to 1 smoothly as τ increases as compare to the profiles of $\bar{\varepsilon} = 0.01$ and $\bar{\varepsilon} = 0.001$. The smoothness in the profiles for small values of $\bar{\varepsilon}$ can be obtained by increasing the number of mesh points. The 3D behaviour of $y(\xi, \tau)$ for different values of $\bar{\varepsilon}$ is shown in Figure 4.6a to 4.6c.

In Table 1, the $\bar{\varepsilon}$-uniform rate of convergence is shown. It is observed that the $\bar{\varepsilon}$-uniform rate of convergence varies from 0.98422 to 0. 99567 for 8 to 64 mesh points, respectively.

Problem 7.2

Consider linear advection-diffusion equation with mixed boundary conditions and

$$\bar{\alpha}(\xi)=\bar{\beta}(\xi)=f(\xi)=1.$$

$$\frac{\partial y}{\partial \tau}=\bar{\varepsilon}\frac{\partial^2 y}{\partial \xi^2}-\frac{\partial y}{\partial \xi}-y+1 \qquad (23)$$

Initial and boundary conditions remain same as of the Problem 7.1. In Figure 6, the solution profiles are shown at different time intervals for $\bar{\varepsilon}=2^{-4}$. In Figure 7, the behaviour of solution profiles is shown for different values of $\bar{\varepsilon}$ at $\xi = 1$ for 32 mesh points. The graphs are found to be approaching to zero smoothly as time increases.

In Table 2, the $\bar{\varepsilon}$-uniform rate of convergence is shown for Problem 7.2. It is observed that the $\bar{\varepsilon}$-uniform rate of convergence is 0.96893 to 0. 99589 for 8 to 64 mesh points, respectively.

8. CONCLUSIONS

In this paper, we study an efficient, accurate, convergent and stable scheme for solving the linear and non linear boundary value problem. Numerical procedure was based on the Cubic Hermite interpolating polynomials, in conjunction with HCM. The numerical results indicate that the proposed scheme provides accurate, stable and convergent for any value of $\bar{\varepsilon}$. Numerical examples are included to demonstrate the validity and applicability of the proposed algorithms.

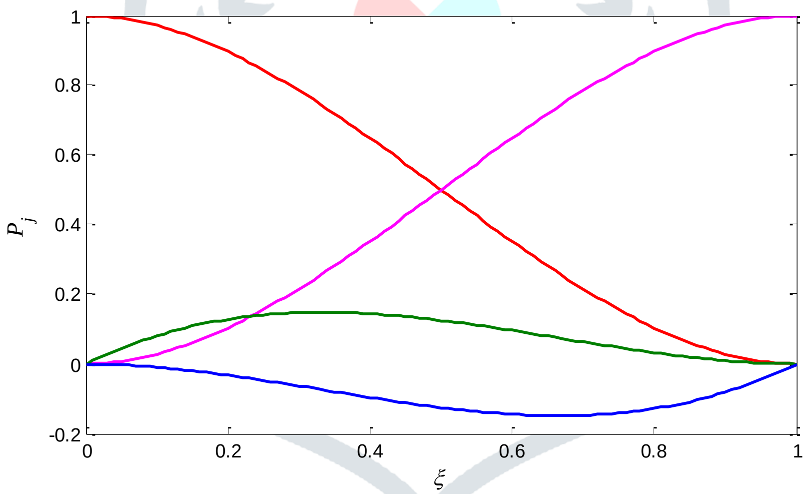

Figure 1: Behavior of Hermite cubics on the interval [ξ_{i-1} ξ_{i+1}] $P_0(\xi)$ (—), $\bar{P}_0(\xi)$ (◊), $P_1(\xi)$ (□), $\bar{P}_1(\xi)$ (Δ).

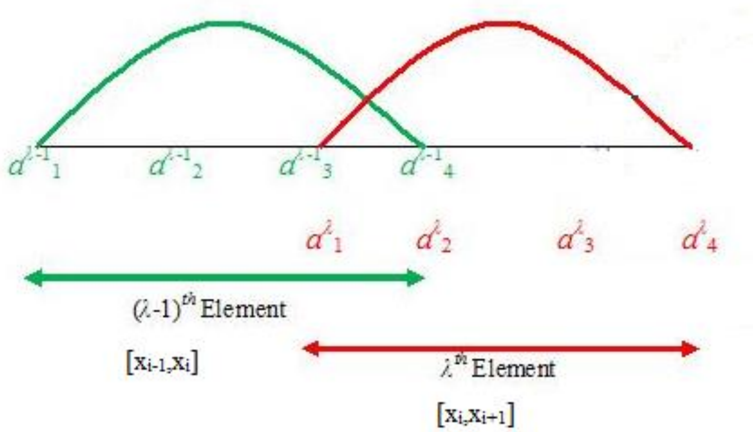

Figure 2: Structure of local Hermite collocation in [ξ_{i-1}, ξ_i] and [ξ_i, ξ_{i+1}].

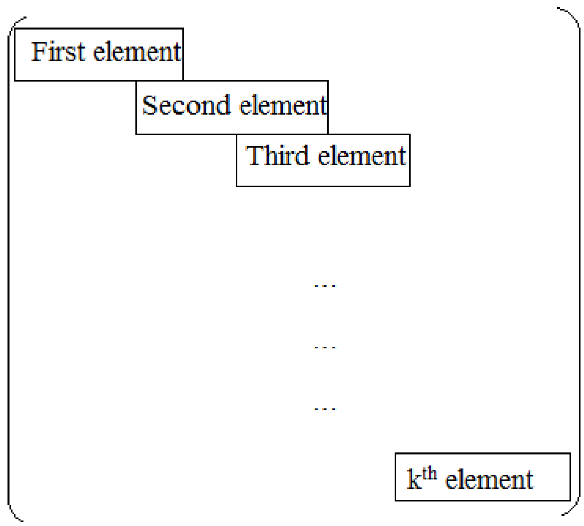

Figure 3: Each block consists of system of linear differential equations defined at j^{th} collocation point in the sub-interval $[\xi_{i-1}\ \xi_i]$.

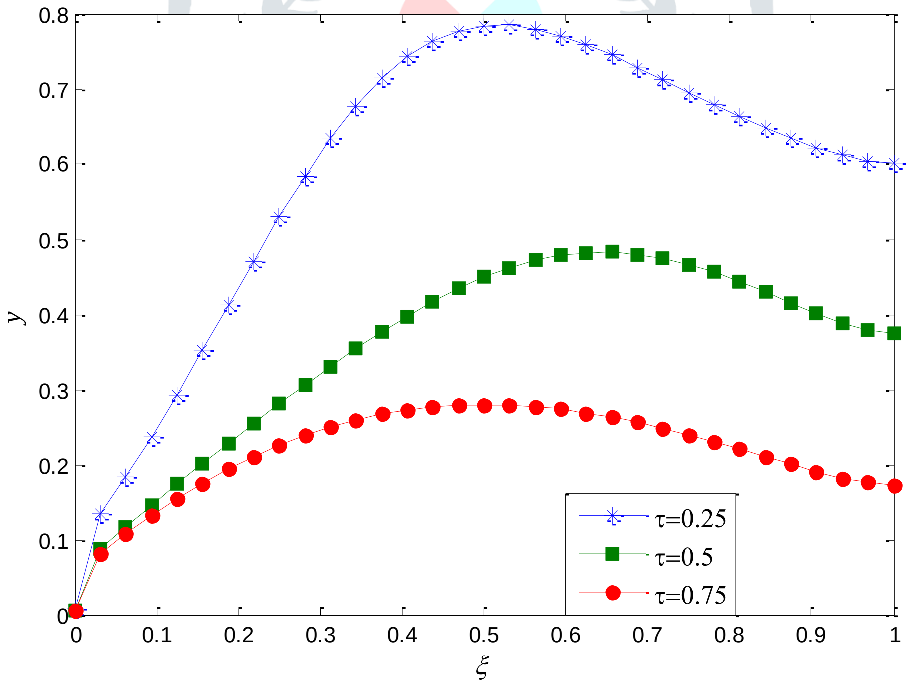

Figure 4: Behavior of y(ξ, τ) for $\bar{\varepsilon} = 2^{-4}$ at different time intervals for Problem 7.1.

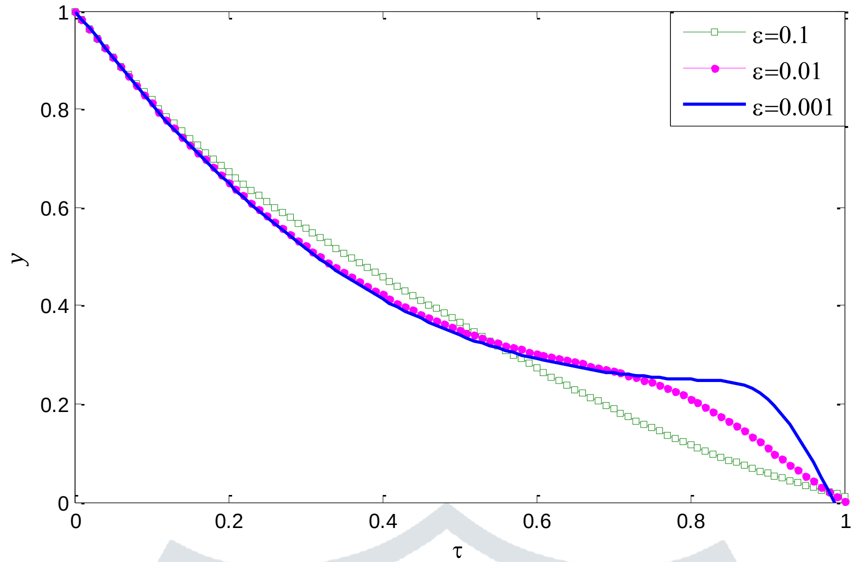

Figure 5: Behavior of y(ξ,τ) for different values of $\bar{\varepsilon}$.

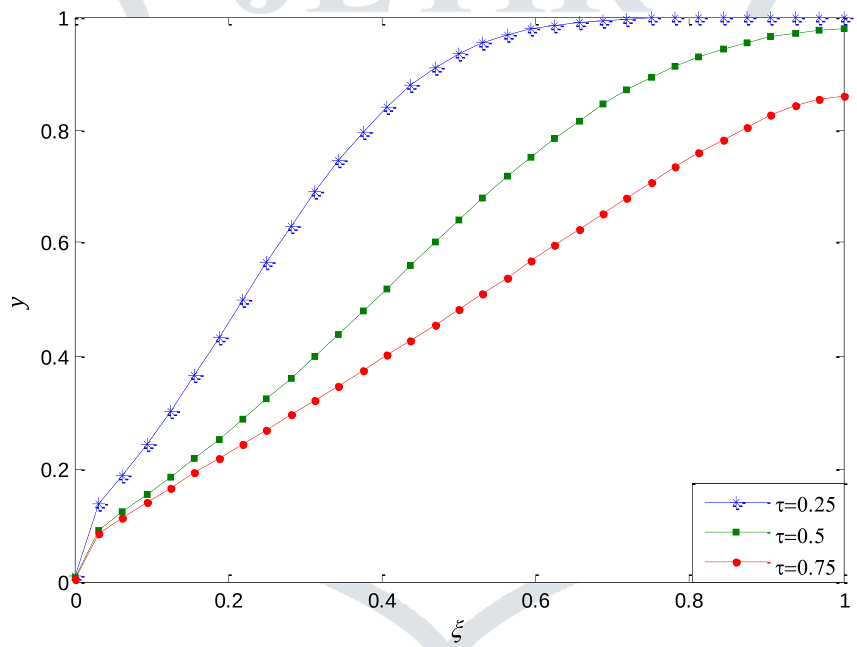

Figure 6: Behavior of y(ξ, τ) for $\bar{\varepsilon} = 2^{-4}$ at different time intervals for Problem 7.2.

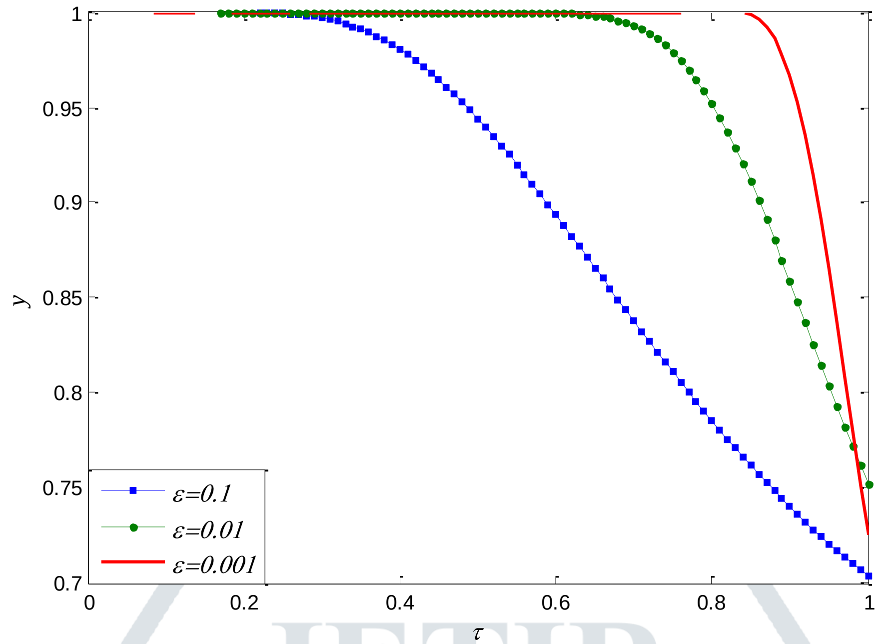

Figure 7: Behavior of y(ξ, τ) for different values of $\bar{\varepsilon}$.

Table 1- $\bar{\varepsilon}$ -uniform error analysis for Problem 7.1.

	$E_{\bar{\varepsilon}}^m$				
$\bar{\varepsilon}$	m = 8	m = 16	m = 32	m = 64	m = 128
2^0	1.6900×10^{-3}	4.0800×10^{-4}	1.0020×10^{-4}	2.4800×10^{-5}	6.1000×10^{-6}
2^{-2}	2.4920×10^{-2}	1.2597×10^{-2}	6.3530×10^{-3}	3.1940×10^{-3}	1.6018×10^{-3}
2^{-4}	1.3042×10^{-2}	4.2572×10^{-3}	1.8113×10^{-3}	8.6330×10^{-4}	4.2660×10^{-4}
2^{-6}	3.0938×10^{-3}	1.3550×10^{-4}	8.1000×10^{-6}	2.0000×10^{-7}	1.0000×10^{-7}
2^{-8}	2.3303×10^{-3}	3.6600×10^{-4}	3.7600×10^{-5}	2.1000×10^{-6}	1.0000×10^{-7}
2^{-10}	1.3851×10^{-3}	3.9390×10^{-4}	9.5600×10^{-5}	9.3000×10^{-6}	3.0000×10^{-7}
2^{-12}	1.1296×10^{-3}	3.5750×10^{-4}	1.1560×10^{-4}	2.4900×10^{-5}	1.9000×10^{-6}
2^{-14}	1.0665×10^{-3}	3.4640×10^{-4}	1.0930×10^{-4}	3.5100×10^{-5}	5.9000×10^{-6}
2^{-16}	1.0508×10^{-3}	3.4380×10^{-4}	1.0580×10^{-4}	3.5900×10^{-5}	9.9000×10^{-6}
2^{-18}	1.0470×10^{-3}	3.4310×10^{-4}	1.0480×10^{-4}	3.5300×10^{-5}	1.5100×10^{-5}
2^{-20}	1.0460×10^{-3}	3.4300×10^{-4}	1.0450×10^{-4}	3.5100×10^{-5}	1.1800×10^{-5}
E^m	2.4920×10^{-2}	1.2597×10^{-2}	6.3530×10^{-3}	3.1940×10^{-3}	1.6018×10^{-3}
p^m	9.8422×10^{-1}	9.8757×10^{-1}	9.9207×10^{-1}	9.9567×10^{-1}	

Table 2- $\bar{\varepsilon}$ -uniform error analysis for Problem 7.2.

	$E_{\bar{\varepsilon}}^m$				
$\bar{\varepsilon}$	m = 8	m = 16	m = 32	m = 64	m = 128
2^0	1.4702×10^{-3}	3.5300×10^{-4}	8.6800×10^{-5}	2.1600×10^{-5}	5.3000×10^{-6}
2^{-2}	2.5127×10^{-2}	1.2837×10^{-2}	6.4884×10^{-3}	3.2627×10^{-3}	1.6360×10^{-3}
2^{-4}	2.2729×10^{-2}	8.1878×10^{-3}	3.6369×10^{-3}	1.7576×10^{-3}	8.7150×10^{-4}
2^{-6}	1.2406×10^{-3}	9.5400×10^{-5}	2.9600×10^{-5}	1.0800×10^{-5}	4.7000×10^{-6}
2^{-8}	3.7452×10^{-3}	6.8500×10^{-5}	2.6000×10^{-6}	1.0000×10^{-7}	0.0000×10^{0}
2^{-10}	3.3890×10^{-3}	5.1600×10^{-5}	8.8000×10^{-6}	7.0000×10^{-7}	1.0000×10^{-7}
2^{-12}	3.1610×10^{-3}	8.2000×10^{-5}	2.1500×10^{-5}	4.1000×10^{-6}	3.0000×10^{-7}

2^{-14}	3.0930×10^{-3}	8.1000×10^{-5}	4.2900×10^{-5}	5.4000×10^{-6}	1.2000×10^{-6}
2^{-16}	3.0760×10^{-3}	7.9000×10^{-5}	5.0100×10^{-5}	6.5000×10^{-6}	7.0000×10^{-7}
2^{-18}	3.0700×10^{-3}	7.9000×10^{-5}	5.2500×10^{-5}	7.0000×10^{-6}	4.0000×10^{-7}
2^{-20}	3.0700×10^{-3}	7.9000×10^{-5}	5.2700×10^{-5}	7.3000×10^{-6}	3.0000×10^{-7}
E^m	2.5127×10^{-2}	1.2837×10^{-2}	6.4884×10^{-3}	3.2627×10^{-3}	1.6360×10^{-3}
p^m	9.6893×10^{-1}	9.8437×10^{-1}	9.9180×10^{-1}	9.9589×10^{-1}	

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