ANALYSIS OF A FRACTIONAL ORDER SIR MODEL

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Abstract :This paper investigates the nonlinear interactions of a fractional order SIR model. The equilibrium points are computed for the fractional order system and stability of the equilibrium points are analyzed. Numerical simulations are used to study the Complex dynamical behaviors of the system. Analytical results are illustrated with numerical simulations. In particular, the effect of fractional order is explored.

***IndexTerms* - Fractional order, SIR Model, Stability, Generalized Euler Method, Bifurcation.**

I. INTRODUCTION

In this decade, study of the dynamics of fractional order systems has become a prominent area of research. Fractional calculus has found applications in nearly all kinds of fields. The ability of fractional calculus to deal with systems with memory makes it a very suitable tool for the description of memory and hereditary properties of systems [2, 3, 7]. The fractional order linear system of the stability results are given below.

Lemma 1. *[4] The fractional order autonomous system* $D^{\alpha}x(t) = Ax(t), x(0) = x_0$ *where* $0 < \alpha < 1, x \in R^n$ *and* $A \in R^{n\times n}$ *is*

(a) locally asymptotically stable if and only if $\left|\arg(\lambda_i(A))\right| > \alpha\frac{\pi}{2}$, $(i = 1,2,\ldots,n)$ *where* $\arg(\lambda_i(A))$ *denotes the argument of the eigenvalue* λ_i *of* A. *(b) stable if and only if* $\left|\arg(\lambda_i(A))\right| \geq \alpha\frac{\pi}{2}, (i = 1,2,\ldots,n)$.

II. GENERALIZED TAYLORS FORMULA AND EULER METHOD

The generalization Taylors formula that involves Caputo fractional derivatives. This generalization is presented in [6]. Suppose that $D^{k\alpha}f(x) \in C(0,a]$ for $k = 0,1,\ldots,n+1$, where $0 < \alpha \leq 1$. Then we have

$$f(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) + \frac{(D^{(n+1)\alpha} f)(\zeta)}{\Gamma((n+1)\alpha+1)} x^{(n+1)\alpha}, 0 \leq \zeta \leq x, \forall x \in C(0,a] \tag{1}$$

For $\alpha = 1$, the generalized Taylor's formula (1) reduces to the classical Taylor's formula. Z. M. Odibat and S. Momani derived the Generalized Euler's Method(GEM) from Classical Euler's method for Initial Value Problems (IVP's) with caputo derivatives [5]. For the following general form of IVP:

$$D^{\alpha}y(t) = f(t, y(t)), y(0) = y_0 \tag{2}$$

for $0 < \alpha \leq 1, 0 < t < \alpha$, the general formula for Generalized Euler's Method (GEM) is

$$y(t_{j+1}) = y(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_j, y(t_j)), j = 0,1,\ldots,n-1 \tag{3}$$

For $\alpha = 1$, (3) reduce Generalized Euler's Method to Classical Euler's Method.

III. MODEL DESCRIPTION

The model under discussion is a fractional order SIR epidemic model with vaccination and treatment. The total population $N(t)$ is partitioned into three compartments namely Susceptible Population $S(t)$, Infected Population $I(t)$ and Recovered Population $R(t)$. In 1927, Kermack and McKendrick established the classical SIR model for the transmission of infectious disease [1]. The system of fractional order equations under discussion is

$$\begin{aligned} D^{\alpha}S(t) &= b - \beta S(t)I(t) - \mu_1 S(t) \\ D^{\alpha}I(t) &= \beta S(t)I(t) - (\mu_1 + \mu_2 + \delta)I(t) \\ D^{\alpha}R(t) &= \delta I(t) - \mu_1 R(t) \end{aligned} \tag{4}$$

where $0 < \alpha < 1$ and the initial values are $S(0), I(0)$ and $R(0)$.

The parameters $b, \beta, \mu_1, \mu_2, \delta$ assume positive values and $S(t), I(t), R(t)$ denote the number of the individuals susceptible to the disease, number of infected members and number of members who have been removed from the possibility of infection through full immunity. b (the birth) is the influx of individual into the susceptible, μ_1 is the natural death rate and individuals in $I(t)$ suffer an additional death due to disease with rate μ_2, β is the disease transmission coefficient and δ is the rate of recovery from infection. Since $D^\alpha N(t) = b - \mu_1 N(t) - \mu_2 I(t)$, System (4) is expressed as

$$\begin{aligned} D^\alpha S &= b - \beta SI - \mu_1 S \\ D^\alpha I &= \beta SI - (\mu_1 + \mu_2 + \delta)I \\ D^\alpha N &= b - \mu_1 N - \mu_2 I \end{aligned} \qquad (5)$$

where $0 < \alpha < 1$ and the initial values are $S(0), I(0)$ and $N(0)$.

IV. BASIC REPRODUCTIVE NUMBER, EQUILIBRIUM POINTS AND STABILITY

Next Generation Matrix (NGM) Approach can be employed to determine R_0. The Next Generation Matrix is given by $K = FV^{-1}$, where $F = \begin{bmatrix} \beta S & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} \mu_1 + \mu_2 + \delta & -\beta I \\ \beta S & \beta I + \mu_1 \end{bmatrix}$. Hence $K = FV^{-1} = \begin{bmatrix} \dfrac{\beta b}{\mu_1(\mu_1 + \mu_2 + \delta)} & 0 \\ 0 & 0 \end{bmatrix}$

Since R_0 is the most dominant eigenvalue of Next Generation Matrix, we obtain $R_0 = \dfrac{\beta b}{\mu_1(\mu_1 + \mu_2 + \delta)}$. Equilibrium points of the fractional order system (5) are evaluated from $D^\alpha S = 0, D^\alpha I = 0, D^\alpha N = 0$. We obtain the following Equilibrium Points

1. Disease Free Equilibrium Point $E_0 = \left(\dfrac{b}{\mu_1}, 0, \dfrac{b}{\mu_1} \right)$
2. Endemic Equilibrium point $E_1 = \left(\dfrac{\mu_1 + \mu_2 + \delta}{\beta}, \dfrac{b}{\mu_1 + \mu_2 + \delta} - \dfrac{\mu_1}{\beta}, \dfrac{b}{\mu_1} - \dfrac{\mu_2 b}{\mu_1(\mu_1 + \mu_2 + \delta)} + \dfrac{\mu_2}{\beta} \right)$.

The dynamical behavior of system around each equilibrium point can be studied by investigating the local stability of the equilibrium point. The local stability is determined by the eigenvalues of the Jacobian matrix evaluated at the equilibrium point.

Theorem 2.*Disease Free Equilibrium is locally asymptotically stable if* $0 < R_0 < 1$ *and unstable if* $R_0 > 1$.

Proof. Based on (5), the Jacobian matrix is

$$J(S, I, N) = \begin{bmatrix} -\beta I - \mu_1 & -\beta S & 0 \\ \beta I & \beta S - (\mu_1 + \mu_2 + \delta) & 0 \\ 0 & -\mu_2 & -\mu_1 \end{bmatrix} \qquad (6)$$

Jacobian Matrix of the Disease Free Equilibrium E_0 is

$$J(E_0) = \begin{bmatrix} -\mu_1 & -\beta \dfrac{b}{\mu_1} & 0 \\ 0 & \beta \dfrac{b}{\mu_1} - (\mu_1 + \mu_2 + \delta) & 0 \\ 0 & -\mu_2 & -\mu_1 \end{bmatrix}$$

The eigen values are $\lambda_1 = \lambda_2 = -\mu_1$ and $\lambda_3 = \beta \dfrac{b}{\mu_1} - (\mu_1 + \mu_2 + \delta)$.

When $0 < R_0 < 1$ and $|\arg(\lambda_{1,2,3})| > \alpha \dfrac{\pi}{2}$, the Disease Free Equilibrium point of the system (5) is asymptotically stable.

Theorem 3.*The system of equations (5) has an endemic equilibrium point and locally asymptotically stable if* $R_0 > 1$.

Proof: At the Endemic Equilibrium E_1, the Jacobian matrix is

$$J(E_1) = \begin{bmatrix} \dfrac{-\beta b}{\mu_1 + \mu_2 + \delta} & -(\mu_1 + \mu_2 + \delta) & 0 \\ \dfrac{\beta b}{\mu_1 + \mu_2 + \delta} - \mu_1 & 0 & 0 \\ 0 & -\mu_2 & -\mu_1 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = -\mu_1$ and $\lambda_{2,3} = \frac{-\beta b}{2(\mu_1+\mu_2+\delta)} \pm \frac{1}{2}\sqrt{\left(\frac{\beta b}{\mu_1+\mu_2+\delta}\right)^2 + 4\left[(\beta b) - \mu_1(\mu_1+\mu_2+\delta)\right]}.$

When $R_0 > 1$ and $\left|\arg(\lambda_{1,2,3})\right| > \alpha\frac{\pi}{2}$, the Endemic Equilibrium point of the system (5) is asymptotically stable.

V. NUMERICAL EXAMPLES

Numerical solution of the fractional order system is

$$S(t_{j+1}) = S(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(b - \beta S(t_j)I(t_j) - \mu_1 S(t_j)\right)$$

$$I(t_{j+1}) = I(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(\beta S(t_j)I(t_j) - (\mu_1+\mu_2+\delta)I(t_j)\right)$$

$$N(t_{j+1}) = N(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(b - \mu_1 N(t_j) - \mu_2 I(t_j)\right)$$

for $j = 0,1,\ldots,k-1.$

Numerical simulations are performed to support analytical results of the model. In the following simulations, initial conditions are taken as $\left(S(0), I(0), N(0)\right) = \left(0.95, 0.05, 1.00\right)$.

Example 1.*Consider the parameter values* $b = 0.2, \mu_1 = 0.2, \mu_2 = 0.2, \beta = 0.1, \delta = 0.3, h = 0.1$ *with the fractional derivative order* $\alpha = 0.9$. *The corresponding eigenvalues are* $\lambda_1 = -0.2$ *and* $\lambda_2 = -0.6$ *for* E_0. *Also* $\left|\arg(\lambda_{1,2})\right| = 3.1416 > 1.4137 = \alpha\frac{\pi}{2}$ *and* R_0

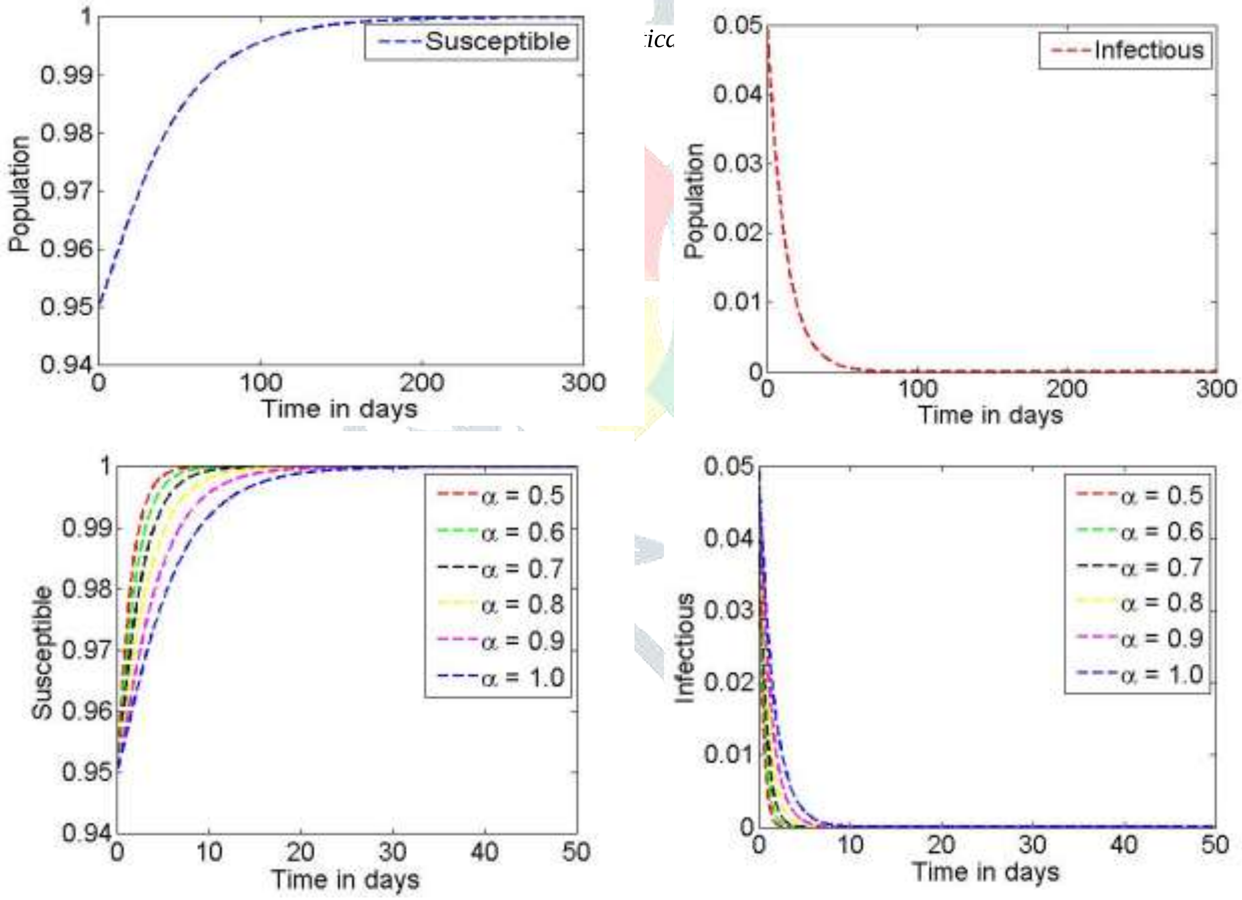
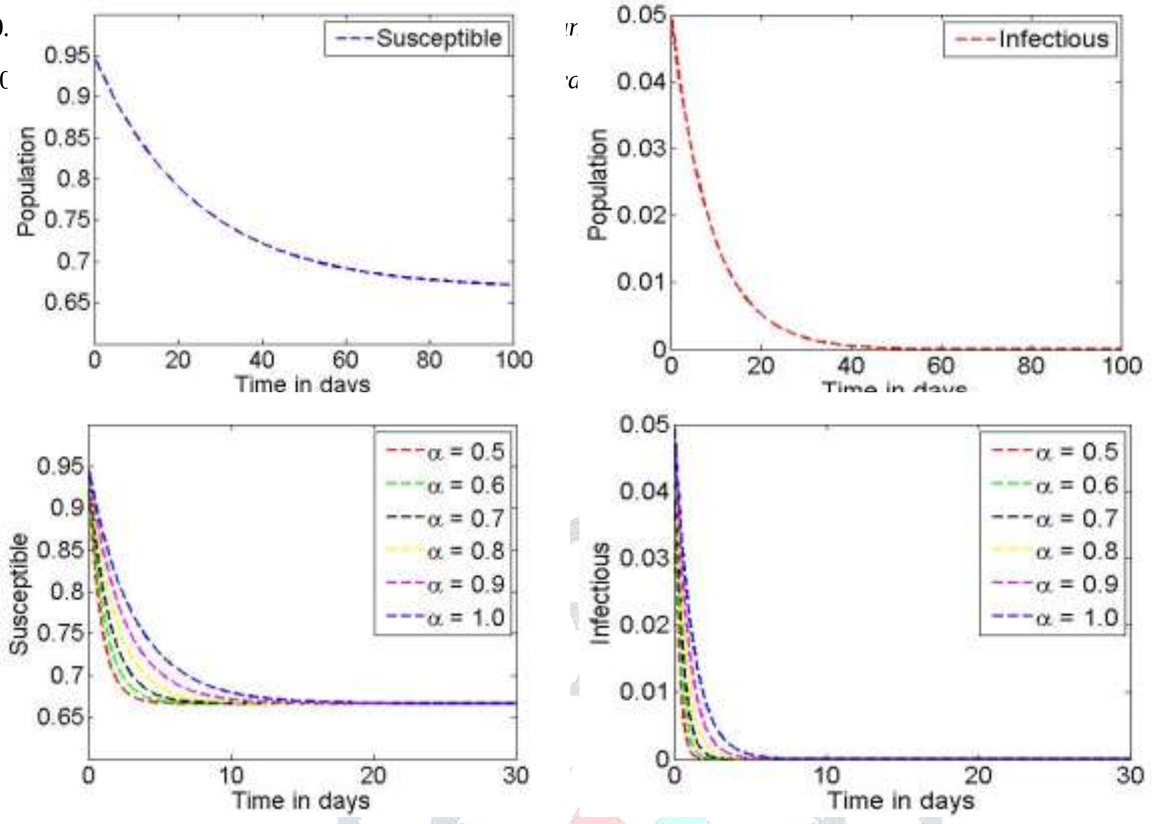


Figure 1. Time series of disease free equilibrium E_0 *and Different Fractional Derivatives* $(\alpha's)$ *with Stability of* $R_0 < 1$.

Example 2. *Consider the parameter values* $b=0.2, \mu_1=0.3, \mu_2=0.3, \beta=0.1, \delta=0.3, h=0.1$ *and the fractional derivative order*

$\alpha=0.$ *r* $137=\alpha\frac{\pi}{2}$ *and*

$R_0=($ *a*



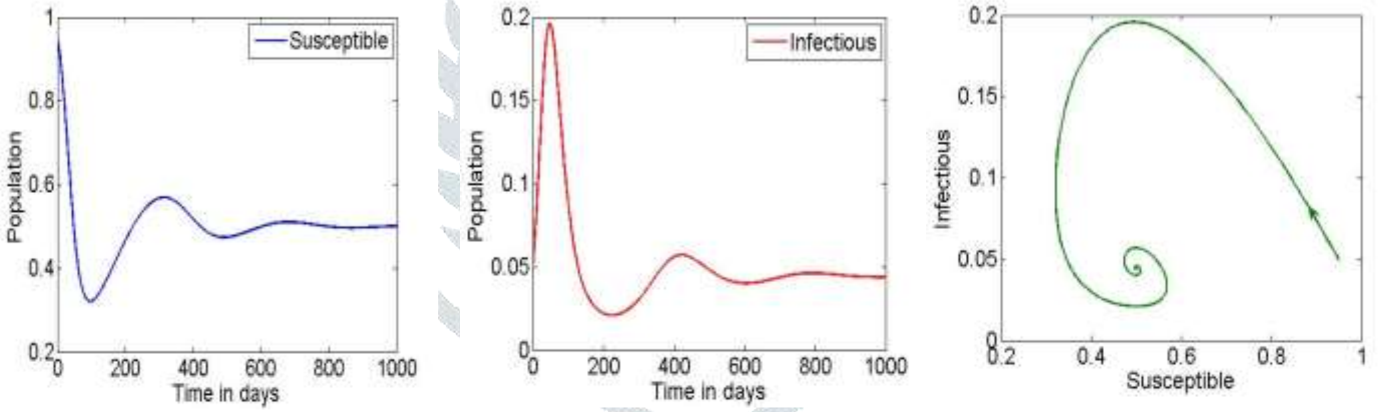


Figure 2. Time series of disease free equilibrium E_0 and Different Fractional Derivatives $(\alpha's)$ with Stability of $R_0<1$

Example 3. *For the parameter values* $b=0.04, \mu_1=0.04, \mu_2=0.4, \beta=0.9, \delta=0.01, h=0.1$ *and the fractional derivative order* $\alpha=0.9$. *The corresponding eigenvalues are* $\lambda_1=-0.04$ *and* $\lambda_{2,3}=-0.0400\pm0.1281i$ *for* E_1. *Here* $|\arg(\lambda_{1,2,3})|=3.1416>1.4137=\alpha\frac{\pi}{2}$ *and* $R_0=2>1$. *Hence the endemic equilibrium is asymptotically stable (Figure 3)*

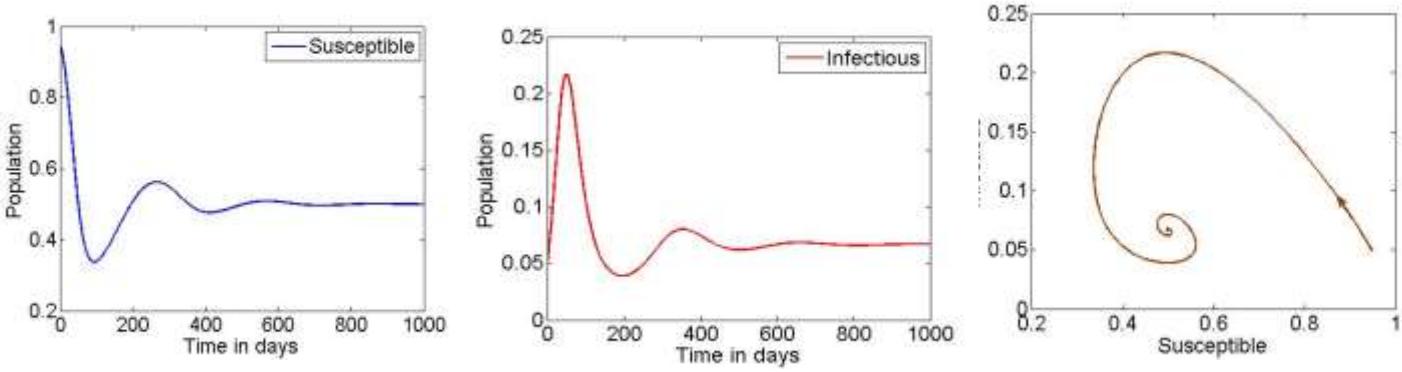
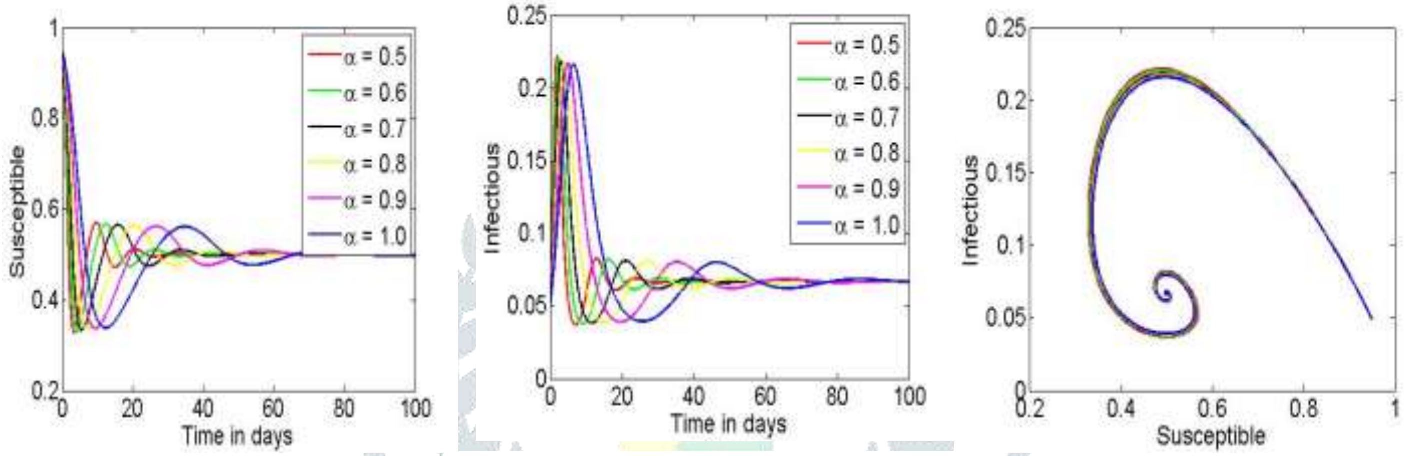
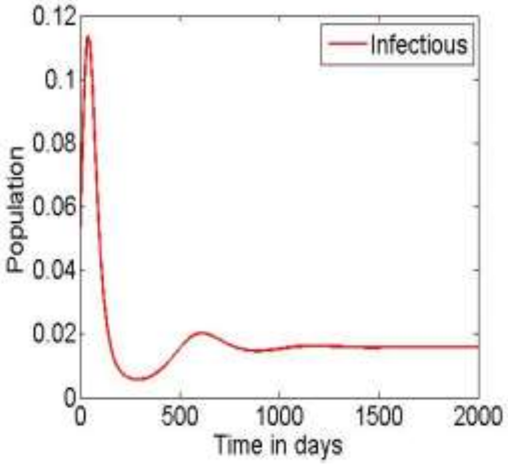
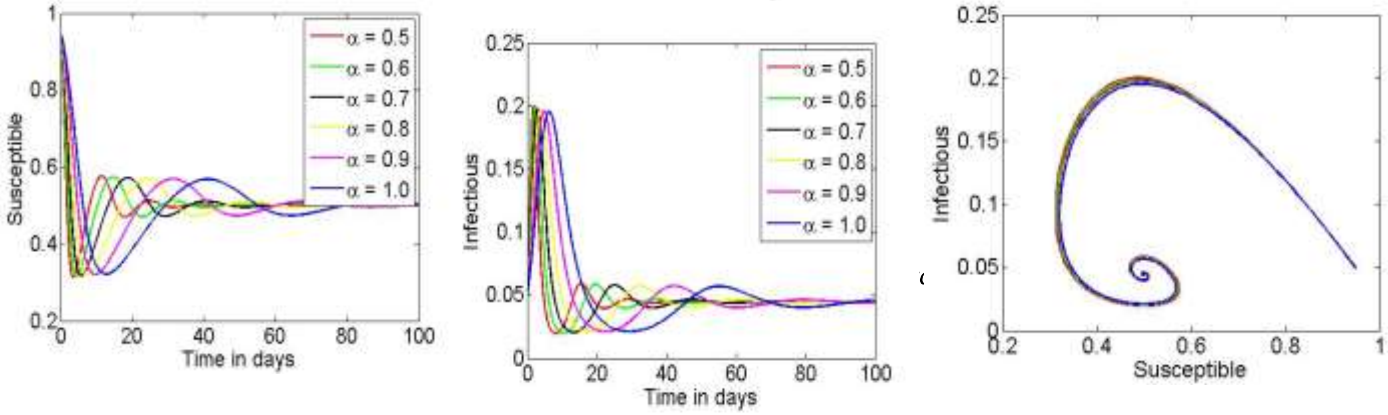


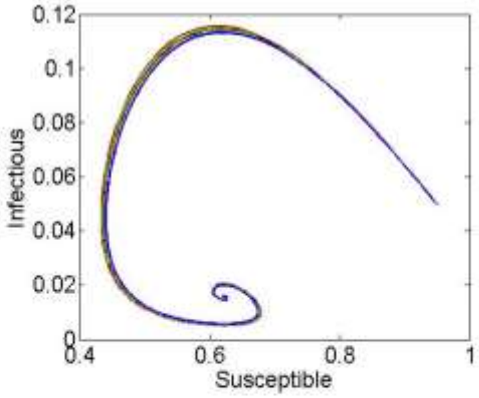
Figure 3.Time series and Phase diagram of endemic equilibrium E_1 and Different Fractional Derivatives $(\alpha's)$ with Stability of $R_0 > 1$.

Example 4.*Let us consider the parameter values $b = 0.05, \mu_1 = 0.04, \mu_2 = 0.4, \beta = 0.9, \delta = 0.01, h = 0.1$ and the fractional derivative order $\alpha = 0.9$. For these parameters the corresponding eigenvalues are $\lambda_1 = -0.04$ and $\lambda_{2,3} = -0.0500 \pm 0.1565i$ for E_1. If $\left|\arg(\lambda_{1,2,3})\right| = 3.1416 > 1.4137 = \alpha\dfrac{\pi}{2}$ and $R_0 = 2.5 > 1$. Then the endemic equilibrium is asymptotically stable See (Figure 4.)*









$R_0 > 1$.

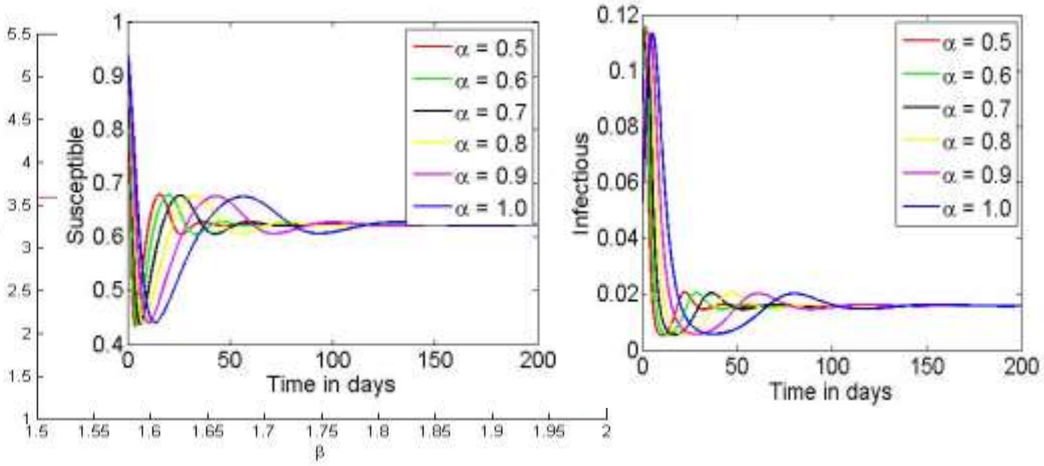
Example 5.*Consider the parameter values* $b = 0.04, \mu_1 = 0.05, \mu_2 = 0.5, \beta = 0.9, \delta = 0.01, h = 0.1$ *and the fractional derivative order* $\alpha = 0.9$. *For these parameters the corresponding eigenvalues are* $\lambda_1 = -0.05$ *and* $\lambda_{2,3} = -0.0321 \pm 0.0835i$ *for* E_1. *If*

$\left|\arg(\lambda_{1,2,3})\right| = 3.1416 > 1.4137 = \alpha\frac{\pi}{2}$ *and* $R_0 = 1.2857 > 1$ *.So the endemic equilibrium is asymptotically stable See (Figure 5.)*

Figure 5. Time series and Phase diagram of endemic equilibrium E_1 *and Different Fractional Derivatives* $(\alpha's)$ *with Stability of* $R_0 > 1$.

IV. BIFURCATION

The following bifurcation diagrams exhibit the change in qualitative behaviour of the system (5). Stability and periodic orbits with the birth of limit cycles can be identified. Bifurcation diagrams are produced by varying the control parameter



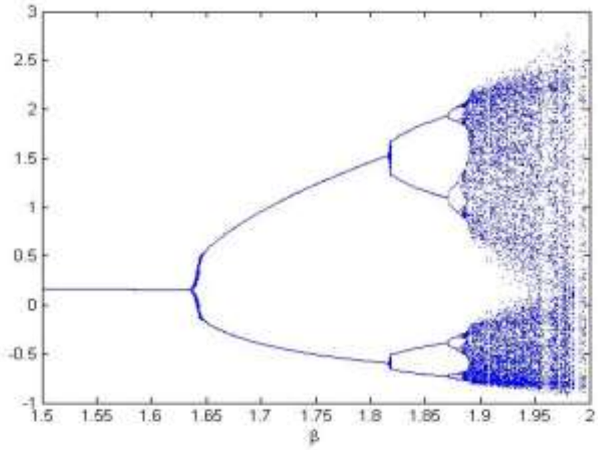


Figure 6. Bifurcation of Susceptible population, Infected population with initial values
$(S_0, I_0) = (0.95, 0.05), \beta \in [1.5, 2.0], b = 0.9, \mu_1 = 0.02, \mu_2 = 0.03, h = 0.1, \delta = 0.2, \alpha = 0.5$

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