ANALYSIS OF DISCRETE FRACTIONAL ORDER PREY POPULATION SYSTEM WITH ALLEE EFFECT ON PREY POPULATION

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Abstract :This paper investigates the dynamical behavior of a Fractional Order Prey – Predator system in the presence of Allee Effect. A discretization process will be applied to obtain its discrete version. The fixed points and the asymptotic stability are investigated. Also limit cycles and bifurcation diagram is provided for selected range of growth parameter. We also show that the positive fixed point changes from stable to unstable and unstable to stable under the Allee Effect on prey population.

***IndexTerms* - Fractional Order, Discretization, Fixed Point, Allee Effect, Local Stability, Limit Cycles, Bifurcations.**

I. INTRODUCTION

Mathematical modeling of interactions between species has drawn the attention of researchers [3],[9]. Population models in ecology have been studied using ordinary differential equations, difference equations, partial differential equations, fractional order differential equations and stochastic models. Fractional Order Differential Equations (FODE) are suitable to study the systems with memory which exists in most biological systems. Stability and dynamical analysis of fractional order LotkaVolterra Models can be found in [8], [12].

The investigation of the population dynamics with Allee effect has attracted the researchers in recent decades. Allee effect describes a positive relation between population density and the per capita growth rate. This effect can be caused by difficulties in, for example, mate finding, social dysfunction at small population sizes and predator avoidance of defense. Allee effect, where fitness is reduced when conspecific density is low, can dramatically affect the dynamics of species interactions [1], [4], [5], [6].In this paper, we study the dynamical behaviors of fractional order LotkaVolterra predator prey system subject to the Allee effect on prey population.

II. ALLEE EFFECT ON PREY POPULATION

In this paper, we consider the following fractional order prey-predator system subject to an Allee effect on prey population:

$$\begin{aligned} D^{\alpha}x(t) &= rx(t)[1-x(t)]\frac{x(t)}{a+x(t)} - bx(t)y(t) \\ D^{\alpha}y(t) &= cx(t)y(t) - dy(t) \end{aligned} \qquad (1)$$

Here we take $\frac{x(t)}{a+x(t)}$ as an Allee function and a as an Allee constant. Where $t>0$ and α is the fractional order satisfying $\alpha \in (0,1]$. Now, applying the discretization process for a fractional-order system described in [2,10], we obtain the discrete fractional order Allee effect predator prey system as follows:

$$\begin{aligned} x_{t+1} &= x_t + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\left(rx_t(1-x_t)\frac{x_t}{a+x_t} - bx_t y_t \right) \\ y_{t+1} &= y_t + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\left(cx_t y_t - dy_t \right) \end{aligned} \qquad (2)$$

III. STABILITY OF FIXED POINTS

The fixed points are $E_0=(0,0), E_1=(1,0), E_2=\left(\frac{d}{c}, \frac{rd(c-d)}{bc(ac+d)}\right)$. We next study the local stability of the fixed points.

The Jacobian matrix J of model (1) evaluated at any fixed point (x^*, y^*) is given by

$$J(x^*, y^*) = \begin{bmatrix} \frac{rx^*[2a-(3a-1)x^* - 2x^{*2}]}{(a+x^*)^2} - by^* & -bx^* \\ cy^* & cx^* - d \end{bmatrix}.$$

Theorem 1.*The extinction fixed point E_0 is locally asymptotically stable if $d<1$, otherwise unstable fixed point.*

Proof: The Jacobian matrix for the extinction fixed point E_0 is $J(E_0)=\begin{bmatrix}0 & 0\\ 0 & -d\end{bmatrix}$.

Hence the eigenvalues of $J(E_0)$ are $\lambda_1=0$ and $\lambda_2=-d$. Thus E_0 is stable when $d<1$. Otherwise E_0 is unstable fixed point.

Theorem 2.*The exclusion fixed point E_1 is locally asymptotically stable if $r<1+a$ and $d<1+c$, otherwise unstable fixed point.*

Proof: The Jacobian matrix for the exclusion fixed point E_1 is

$$J(E_1)=\begin{bmatrix}-\dfrac{r}{1+a} & -b\\ 0 & c-d\end{bmatrix}.$$

The eigenvalues of the matrix $J(E_1)$ are $\lambda_1=-\dfrac{r}{1+a}$ and $\lambda_2=c-d$. Hence E_1 is locally asymptotically stable when $r<1+a$ and $d<1+c$, unstable when $r>1+a$ and $d>1+c$.

We now focus on the positive fixed point E_2, we have the following coefficient matrix: $J(E_2)=\begin{bmatrix}\beta & -\dfrac{bd}{c}\\ \dfrac{rd(c-d)}{b(ac+d)} & 0\end{bmatrix}$

where $\beta=\dfrac{rx^*[2a-(3a-1)x^*-2x^{*2}]}{(a+x^*)^2}-by^*$. Thus the matrix $J(E_2)$ yields the characteristic equation

$$P(\lambda)=\lambda^2-\beta\lambda+\frac{rd^2(c-d)}{c(ac+d)}=0 \qquad (3)$$

It follows from the well-known *Jury conditions*(See for instance [7]) that the modulus of all roots of equation (3) is less than one if and only if the conditions $P(1)>0, P(-1)>0$ and $\det J(E_2)<1$ holds.

Now, we first obtain that $P(1)>0$ holds if and only if $r>\dfrac{(\beta-1)c(ac+d)}{d^2(c-d)}$, if $c\neq d$. On the other hand $P(-1)>0$ holds if and only if $r<\dfrac{(\beta+1)c(ac+d)}{d^2(d-c)}$,if $d\neq c$. Finally $\det J(E_2)<1$ holds if and only if $r<\dfrac{c(ac+d)}{d^2(c-d)}$, if $c\neq d$.

Theorem 3.*The positive fixed point E_2 of the prey-predator system (1) is locally asymptotically stable if $r<\dfrac{(\beta+1)c(ac+d)}{d^2(d-c)}$, if $d\neq c$ and $r<\dfrac{c(ac+d)}{d^2(c-d)}$, if $c\neq d$ holds.*

The next result follows from Theorem (3) immediately.

Corollary 1.*The positive fixed point E_2 of model (1) is unstable if $r>\dfrac{(\beta+1)c(ac+d)}{d^2(d-c)}$ if $d\neq c$ and $r>\dfrac{c(ac+d)}{d^2(c-d)}$, if $c\neq d$ holds.*

IV. STABILITY ANALYSIS OF DISCRETIZATION FRACTIONAL ORDER

We will now discuss the dynamics of the discretized fractional order LotkaVolterra predator prey model with Allee effect (2). The dynamical behaviors of model (2) is determined by the parameters a,b,c,d,r,s and α . The Jacobian matrix J of model (2) evaluated at any fixed point (x^*,y^*) is given by

$$J(x^*,y^*)=\begin{bmatrix}1+\dfrac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\dfrac{rx^*[2a-(3a-1)x^*-2x^{*2}]}{(a+x^*)^2}\right)-by^* & -\dfrac{s^\alpha}{\alpha\Gamma(\alpha)}bx^*\\ \dfrac{s^\alpha}{\alpha\Gamma(\alpha)}cy^* & 1+\dfrac{s^\alpha}{\alpha\Gamma(\alpha)}\left(cx^*-d\right)\end{bmatrix} \qquad (4).$$

Theorem 4.*If $0<s<\sqrt[\alpha]{\dfrac{2\alpha\Gamma(\alpha)}{d}}$, then the fixed point E_0 is locally asymptotically stable. Otherwise unstable fixed point.*

Proof: The Jacobian matrix J evaluated at the fixed point E_0 has the form $J(E_0)=\begin{bmatrix}1 & 0\\ 0 & 1-\dfrac{s^\alpha}{\alpha\Gamma(\alpha)}d\end{bmatrix}$.

Hence, the eigenvalues of the matrix $J(E_0)$ are $\lambda_1 = 1$ and $\lambda_2 = 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)} d$. Thus E_0 is stable when $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{d}}$.

Otherwise E_0 is unstable when $s > \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{d}}$.

Example 1. *We consider the parameter values* $\alpha = 0.91, s = 0.09, r = 0.09, a = 0.05, b = 0.36, c = 1.17$ *and* $d = 0.12$. *The fixed point* $E_0 = (0,0)$ *and the eigenvalues are* $\lambda_1 = 1$ *and* $\lambda_2 = 0.9861$ *so that* $|\lambda_{1,2}| < 1$. *Hence system (2) is stable. Both prey and predator population will go to extinction (See Figure – 1).*

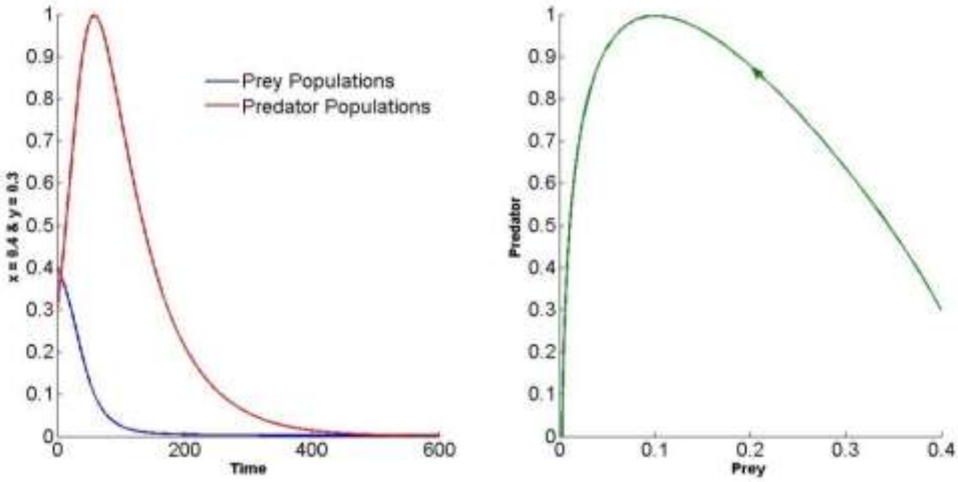

Figure 1. Time series plot and Phase Portrait are Stability at fixed point E_0.

Theorem 5. *The fixed point* E_1 *is locally asymptotically stable if* $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)(1+a)}{r}}$ *and* $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{d-c}}$, *otherwise unstable fixed point.*

Proof: The Jacobian matrix J for the system evaluated at the fixed point E_1 is given by

$$J(E_1) = \begin{bmatrix} 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{r}{1+a}\right) & -\frac{s^\alpha}{\alpha\Gamma(\alpha)} b \\ 0 & 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}(c-d) \end{bmatrix}.$$

Hence, the eigenvalues of the matrix $J(E_1)$ are $\lambda_1 = 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{r}{1+a}\right)$ and $\lambda_2 = 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}(c-d)$. Hence E_1 is locally asymptotically stable when $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)(1+a)}{r}}$ and $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{d-c}}$ and unstable when $s > \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)(1+a)}{r}}$ and $s > \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{d-c}}$.

Example2. *Choosing the parameter values* $\alpha = 0.95, s = 0.61, r = 0.61, a = 0.05, b = 0.01, c = 0.21$ *and* $d = 0.52$. *The fixed point* $E_1 = (1,0)$ *and the eigenvalues are* $\lambda_1 = 0.6294$ *and* $\lambda_2 = 0.8022$ *so that* $|\lambda_{1,2}| < 1$. *We observe the system (2) is stable (SeeFigure-2).*

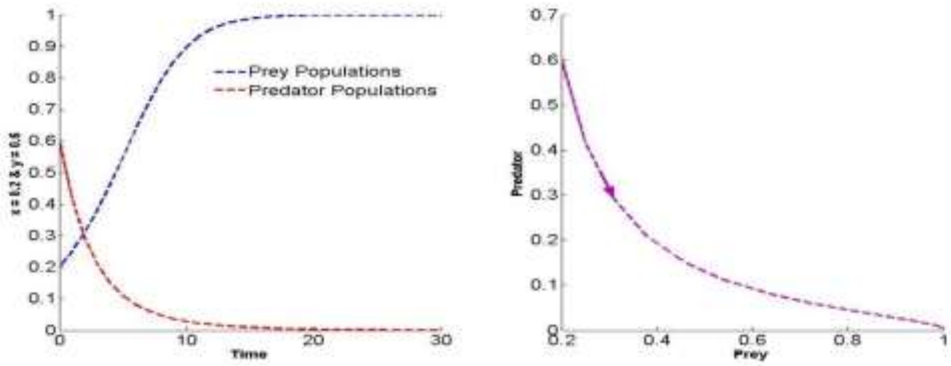

Figure 2. Time series plot and Phase Portrait are Stability at fixed point E_1

Theorem 6.*The positive fixed point E_2 of the prey-predator system (2) is locally asymptotically stable if and only if* $r < \left(\frac{\alpha\Gamma(\alpha)}{s^\alpha}\right)^2 \left(\frac{4 + \frac{s^\alpha}{\alpha\Gamma(\alpha)} 2\beta c(ac+d)}{d^2(d-c)}\right)$ *and* $r > \left(\frac{\alpha\Gamma(\alpha)}{s^\alpha}\right)\left(\frac{\beta c(ac+d)}{d^2(d-c)}\right)$, *if* $d \neq c$ *holds.*

Proof: The Jacobian matrix J evaluated at the positive fixed point E_2 has the form

$$J(E_2) = \begin{bmatrix} 1 - \frac{s^\alpha}{\alpha\Gamma(\alpha)}\beta & -\frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{bd}{c}\right) \\ \frac{s^\alpha}{\alpha\Gamma(\alpha)}\left(\frac{rd(c-d)}{b(ac+d)}\right) & 1 \end{bmatrix}.$$

The trace and determined of the Jacobian matrix $J(E_2)$ are given by

$Tr(J(E_2)) = 2 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\beta$, $Det(J(E_2)) = 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}\beta + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 \left(\frac{rd^2(c-d)}{c(ac+d)}\right)$. Again by using the Jury conditions, we most have that $P(1) > 0$ holds if and only if $r > 0$. Similarly $P(-1) > 0$ and $\det J(E_2) < 1$ holds if and only if $r < \left(\frac{\alpha\Gamma(\alpha)}{s^\alpha}\right)^2 \left(\frac{4 + \frac{s^\alpha}{\alpha\Gamma(\alpha)} 2\beta c(ac+d)}{d^2(d-c)}\right)$ and $r > \left(\frac{\alpha\Gamma(\alpha)}{s^\alpha}\right)\left(\frac{\beta c(ac+d)}{d^2(d-c)}\right)$, if $d \neq c$, respectively.

The next result is obtained from Theorem (6) immediately.

Corollary 2.*The positive fixed point E_2 of system (2) is unstable if* $r > \left(\frac{\alpha\Gamma(\alpha)}{s^\alpha}\right)^2 \left(\frac{4 + \frac{s^\alpha}{\alpha\Gamma(\alpha)} 2\beta c(ac+d)}{d^2(d-c)}\right)$ *and* $r < \left(\frac{\alpha\Gamma(\alpha)}{s^\alpha}\right)\left(\frac{\beta c(ac+d)}{d^2(d-c)}\right)$, *if* $d \neq c$ *holds.*

Example 3.*When* $\alpha = 0.95, s = 0.26, r = 0.26, a = 0.05, b = 1.99, c = 1.44$ *and* $d = 0.37$. *The eigenvalues are* $\lambda_{1,2} = 0.9958 \pm i0.0693$ *and* $E_2 = (0.2569, 0.0812)$ *so that* $|\lambda_{1,2}| = 0.9983 < 1$. *We see that system (2) is stable (See figure -3).*

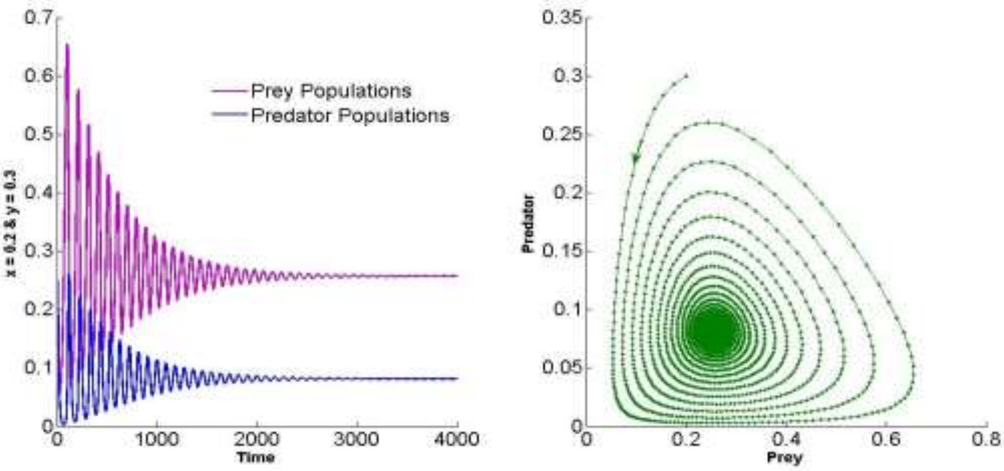

Figure 3. Time series plot and Phase Portrait are Stability at fixed point E_2

V. ALLEE EFFECT INSTABILITY OF COEXISTENCE FIXED POINT

We consider the fractional order of prey-predator model without Allee effect

$$D^\alpha x(t) = rx(t)[1 - x(t)] - bx(t)y(t)$$
$$D^\alpha y(t) = cx(t)y(t) - dy(t) \qquad (5)$$

We obtain the discretization fractional order without Allee effect predator prey system as follows:

$$x_{t+1} = x_t + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\left(rx_t(1-x_t) - bx_t y_t\right)$$
$$y_{t+1} = y_t + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\left(cx_t y_t - dy_t\right) \qquad (6)$$

We illustrate the local stability analysis of discretization fractional order of both Allee effect and without Allee effect in systems (2) and (6). In Figure (4) and (5),we show the trajectories of prey and predator densities in the system studied. Figure (4) and (5)

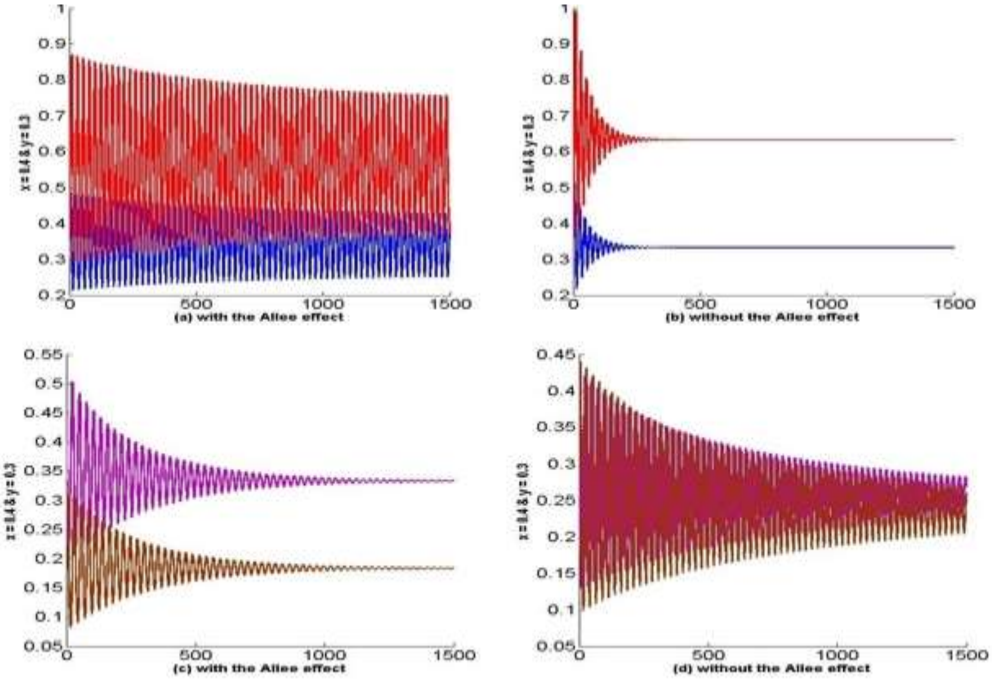

Figure 4. Trajectories of the prey – predator system: the Allee effect stabilize or destabilize system.

presents that when the prey population is subject to an Allee effect, the stability of the positive fixed point change from unstable to stable under the Allee constant as $a = 0.05$. On the other hand, the corresponding fixed point changefrom stable to unstable and

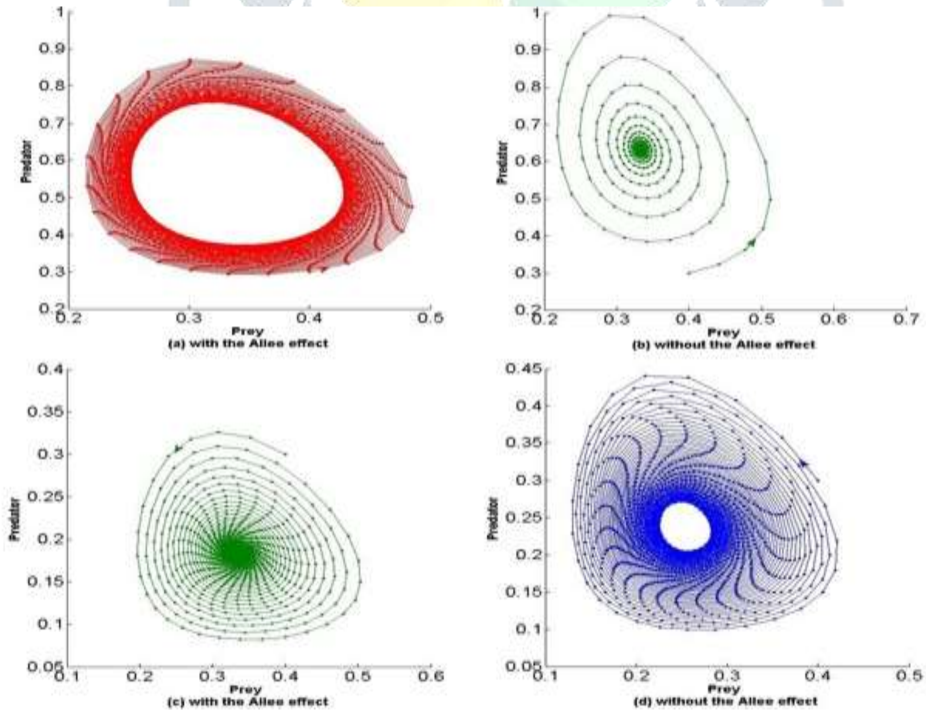

Figure 5.Trajectories of the prey-predator system: the Allee effect stabilize or destabilize the system.

the trajectory spirals inwards but does not approach a point. Finally the trajectory settles down as a limit cycle. Figure 6 presents the bifurcation diagrams of prey and predator densities of the system (2) with initial conditions $x = 0.4$ and $y = 0.3$ as above and the parameter values $\alpha = 0.95, s = 0.65, a = 0.05, b = 1.5, c = 2.6, d = 0.9$ and $r = 2.5 : 0.001 : 5$. Figure (6) shows the bifurcations of prey populations of the system (2), respectively, when the prey population is subject to the Allee effect; however, the other graphs correspond to the bifurcations of predator populations in the prey-predator system (2).

Figure	α	s	r	a	b	c	d	Fixed Point	Initial
4(a),5(a)	0.91	0.58	0.58	0.05	0.61	1.77	0.59	Unstable	(0.4,0.3)
4(b),5(b)	0.91	0.58	0.58	0	0.61	1.77	0.59	Stable	(0.4,0.3)
4(c),5(c)	0.91	0.51	0.51	0.05	1.61	1.77	0.59	Unstable	(0.4,0.3)
4(d),5(d)	0.91	0.51	0.51	0	1.61	2.33	0.59	Stable	(0.4,0.3)

Table 1. Stability Analysis of with Allee and without Allee Effect of the system

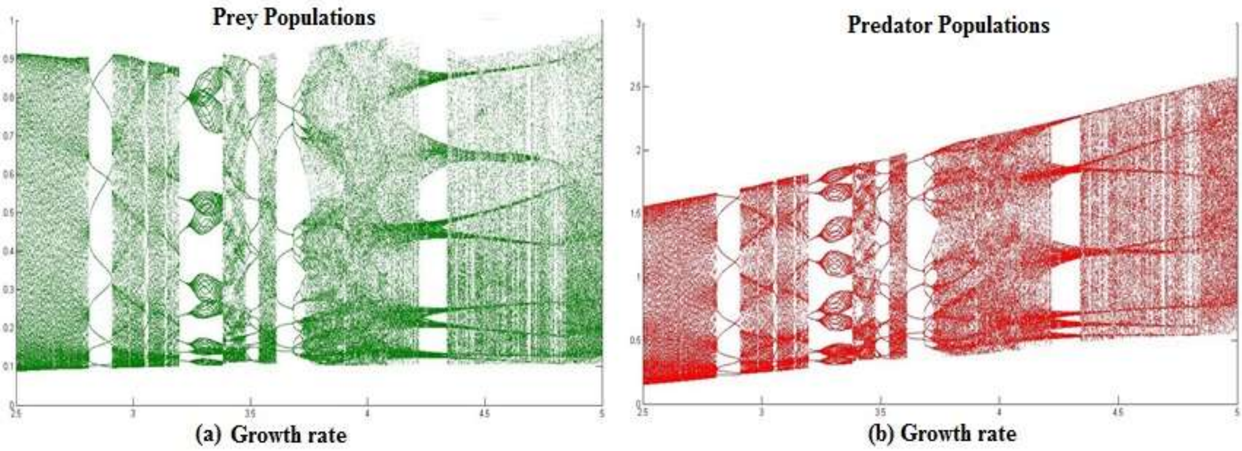

Figure 6. Bifurcation diagrams between prey and predator densities with growth rate in the system (2) respectively.

REFERENCES

[1] Allee, W.C. 1931. Animal Aggretions: A Study in General Sociology.University of Chicago Press,Chicago.

[2] Abd-Elalim A. Elsadany, H. A. EL-Metwally, E. M. Elabbasy, H. N. Agiza, Chaos and bifurcation of anonlinear discrete prey-predator system, Computational Ecology and Software, 2012, 2(3):169-180.

[3]Berrymann, A.A: The origins and evolution of predator-prey theory, Ecology 73, 1530-1535 (1992).

[4]Beddington JR. Age distribution and the stability of simple discrete time population models, J. Theor.Biol. V47, pp. 65-74, 1974.

[5]Birkhead TR. The effect of habitat and density on breeding success in the common guillemot, J. Anim.Ecol. V46, pp. 751-764, 1977.

[6] C. Celik, Stability analysis of a population dynamics model with Allee effect, WCEC, San Francisco,USA, pp. 1-5, October 24-26, 2007.

[7] Murray JD. Mathematical biology, New York: Springer-Verlag; 1993.

[8] Margarita Rivero, Juan J. Trujillo, Luis Vzquez, M. Pilar Velasco, Fractional dynamics of populations,Applied Mathematics and Computation 218 (2011) 1089 - 1095.

[9] Leah Edelstein-Keshet, Mathematical Models in Biology, SIAM, Random House, New York, 2005.

[10] El-Sayed, AMA, El-Mesiry, A, El-Saka, H: On the fractional-order logistic equation. Appl. Math. Lett.20, 817-823 (2007)

[11] El-Sayed, AMA: On the fractional differential equations. J. Appl. Math. Comput. 49(2-3), 205-213(1992)

[12] Ravi P Agarwal, Ahmed MA El-Sayed and Sanaa M Salman, Fractional-order Chuas system: discretization,bifurcation and chaos, Agarwal et al. Advances in Difference Equations 2013, 2013:320.

[13] Saber Elaydi, An Introduction to Difference Equations, ThirdEdition, Springer International Edition,First Indian Reprint, 2008.