qs$\mathcal{I}$- CONNECTEDNESS IN IDEAL BITOPOLOGICAL SPACES

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Abstract:* *The purpose of this paper is to introduce and study the notion of qs$\mathcal{I}$- connectedness in ideal bitopological spaces. We shall also study the notions of qs$\mathcal{I}$- separated sets in ideal bitopological spaces.

Keywords:* *Ideal bitopological spaces, qs$\mathcal{I}$- connected, qs$\mathcal{I}$- separated sets, qs$\mathcal{I}$- s-connected.
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I. INTRODUCTION AND PRELIMINARIES

In 1961 Kelly introduced the concept of bitopological spaces as an extension of topological spaces [8]. A bitopological space (X, τ_1, τ_2) is a nonempty set X equipped with two topologies τ_1 and τ_2 [8]. The study of quasi open sets in bitopological spaces was initiated by Datta in 1971 [1]. In a bitopological space (X, τ_1, τ_2) a set A of X is said to be quasi open if it is a union of a τ_1-open set and a τ_2-open set [1]. Complement of a quasi open set is termed quasi closed. Every τ_1-open (resp. τ_2-open) set is quasi open but the converse may not be true. Any union of quasi open sets of X is quasi open in X. The intersection of all quasi closed sets which contains A is called quasi closure of A. It is denoted by qCl(A) [1]. The union of quasi open subsets of A is called quasi interior of A. It is denoted by qInt(A) [1].

In 1963 Levine [11] introduced the concept of semi open sets in topology. A subset A of a topological space (X, τ) is called semi open if there exists an open set O in X such that $O \subset A \subset Cl(O)$. Every open set is semi open but the converse may not be true. In 1986 Maheshwari, Chae and Thakur [10] generalized this concept by introducing quasi semi- open sets in bitopological spaces. A set A in a bitopological space(X, τ_1, τ_2) is called quasi semi open if it is a union of a τ_1- semi open set and a τ_2- semi open set [10]. Every quasi open (τ_1- semi open, τ_2- semi open) set is quasi semi open but the converse may not be true. A set is said to be quasi semi closed if its complement is quasi semi open. Any union of quasi semi open sets of X is quasi semi open in X. The union of quasi semi open subsets of A is called quasi semi interior of A. The intersection of all quasi semi closed sets which contains A is called quasi semi closure of A. Quasi semi interior and quasi semi closure of A are respectively denoted as qsInt(A) and qsCl(A) [10].

The study of ideal topological spaces was initiated by Vaidyanathaswamy [13] in 1945 and later studied by Kuratowski in 1966 [9]. Applications to various fields were further investigated by Dontchev [2], Jankovic and Hamlett [5] and others.

An Ideal $\mathcal{I}$ on a topological space (X, τ) is a non empty collection of subsets of X which satisfies:
i. $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$ and
ii. $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X, and is denoted by (X, τ, $\mathcal{I}$). If (X) is the set of all subsets of X, in a topological space (X, τ) a set operator $(.)^*$: $\mathcal{P}$ (X) → $\mathcal{P}$ (X) is called the local mapping [2] of A with respect to τ and $\mathcal{I}$ and is defined as follows: $A^*(\tau, \mathcal{I}) = \{x \in X \mid U \cap A \notin \mathcal{I}, \forall\ U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$.

Definition 1.1. [6]. If (X, τ_1, τ_2) is a bitopological space then (X, τ_1, τ_2, $\mathcal{I}$) is an ideal bitopological space if $\mathcal{I}$ is an ideal on X.

In 2010, Jafari and Rajesh defined quasi local mapping of A with respect to τ_1, τ_2 and $\mathcal{I}$ and defined it as follows $A^*_q(\tau_1, \tau_2, \mathcal{I}) = \{x \in X \mid U \cap A \notin \mathcal{I}, \forall$ quasi open set U containing x$\}$ [4].

Definition 1.2. [12]. Given an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) the quasi semi-local mapping of A with respect to τ_1, τ_2 and $\mathcal{I}$ denoted by $A^*_{qs}(\tau_1, \tau_2, \mathcal{I})$ (more generally as A^*_{qs}) is defined as $A^*_{qs}(\tau_1, \tau_2, \mathcal{I}) = \{x \in X \mid U \cap A \notin \mathcal{I}, \forall$ quasi semi-open set U containing x$\}$.

Definition 1.3. [12]. A subset A of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) is qs$\mathcal{I}$- open if $A \subset qsInt(A^*_{qs})$ and qs$\mathcal{I}$- closed if its complement is qs$\mathcal{I}$- open.

Definition 1.4. [12]. A mapping f: (X, τ_1, τ_2, $\mathcal{I}$) → (Y, σ_1, σ_2) is called a qs$\mathcal{I}$- continuous if $f^{-1}(V)$ is a qs$\mathcal{I}$- open set in X for every quasi open set V of Y.

Definition 1.5. [12]. In an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) the quasi $*$ -semi closure of A of X denoted by $qsCl^*(A)$ is defined by $qsCl^*(A) = A \cup A^*_{qs}$

Definition 1.6. [12]. A subset A of an ideal bitopological space (X, $\tau_1, \tau_2, \mathcal{I}$) is said to be a qs$\mathcal{I}$- neighbourhood of a point x ∈ X if ∃ a qs$\mathcal{I}$- open set O such that x ∈ O ⊂ A

Definition 1.7. [12]. Let A be a subset of an ideal bitopological space (X, $\tau_1, \tau_2, \mathcal{I}$) and x ∈ X. Then x is called a qs$\mathcal{I}$-interior point of A if ∃ V a qs$\mathcal{I}$- open set in X such that x ∈ V ⊂ A. The set of all qs$\mathcal{I}$- interior points of A is called the qs$\mathcal{I}$- interior of A and is denoted by qs$\mathcal{I}$Int(A).

Definition 1.8. [12]. Let A be a subset of an ideal bitopological space (X, $\tau_1, \tau_2, \mathcal{I}$) and x ∈ X. Then x is called a qs$\mathcal{I}$-cluster point of A, if V ∩ A ≠ ∅, for every qs$\mathcal{I}$- open set V in X. The set of all qs$\mathcal{I}$-cluster points of A denoted by qs$\mathcal{I}$Cl(A) is called the qs$\mathcal{I}$-closure of A .

Definition 1.9. [3]. An ideal topological space (X, τ, $\mathcal{I}$) is called *-connected if X cannot be written as the disjoint union of a nonempty open set and a nonempty *-open set.

Definition 1.10. [7]. An ideal bitopological space (X, $\tau_1, \tau_2, \mathcal{I}$) is called pairwise *-connected if X cannot be written as the disjoint union of a nonempty τ_i open set and a nonempty τ_j^* -open set. {i , j = 1, 2; i ≠ j}

Definition 1.11. [7]. Nonempty subsets A, B of an ideal bitopological space (X, $\tau_1, \tau_2, \mathcal{I}$), are called pairwise *-separated if τ_iCl*(A) ∩ B = A ∩ τ_jCl(B) = ϕ. {i , j = 1 , 2; i ≠ j}

II. qs$\mathcal{I}$- CONNECTEDNESS IN IDEAL BITOPOLOGICAL SPACES

Definition 2.1. An ideal topological space (X, $\tau_1, \tau_2, \mathcal{I}$) is called qs$\mathcal{I}$- connected if X cannot be written as the disjoint union of a nonempty quasi open set and a nonempty qs$\mathcal{I}$- open set.

Definition 2.2. Nonempty subset A, B of an ideal bitopological space (X, $\tau_1, \tau_2, \mathcal{I}$) are called qs$\mathcal{I}$-separated if qCl(A) ∩ B = A ∩ qs$\mathcal{I}$Cl(B) = ϕ.

Theorem 2.1. If A, B are -separated sets of an ideal bitopological space (X, $\tau_1, \tau_2, \mathcal{I}$) and A ∪ B ∈ $\tau_1 \cap \tau_2$ then A is qs$\mathcal{I}$- open and B is quasi open.

Proof: Since A and B are qs$\mathcal{I}$-separated in X, then B = (A ∪ B) ∩ (X - qCl(A)). Since A∪ B is biopen and qCl(A) is quasi closed in X, B is quasi open in X. Similarly A = (A ∪ B) ∩ (X - qs$\mathcal{I}$Cl(B)) and we obtain that A is qs$\mathcal{I}$-open in X.

Theorem 2.2. Let (X, $\tau_1, \tau_2, \mathcal{I}$) be an ideal bitopological space and A, B ⊂ Y ⊂ X. Then A and B are qs$\mathcal{I}$-separated in Y if and only if A, B are qs$\mathcal{I}$-separated in X.

Proof: It follows from qCl(A) ∩ B = A ∩ qs$\mathcal{I}$Cl(B) = ϕ and the fact that A, B ⊂ Y ⊂ X.

Theorem 2.3. If f: (X, $\tau_1, \tau_2, \mathcal{I}$) → (Y, σ_1, σ_2) is a qs$\mathcal{I}$- continuous onto mapping. Then if (X, $\sigma_1, \sigma_2, \mathcal{I}$) is a qs$\mathcal{I}$-connected ideal bitopological space (Y, σ_1, σ_2) is also quasi connected.

Proof: It is known that connectedness is preserved by continuous surjections. Hence every qs$\mathcal{I}$-open set is also quasi open. Hence, qs$\mathcal{I}$- connected space is also quasi connected.

Definition 2.3. A subset A of an ideal bitopological space (X, $\tau_1, \tau_2, \mathcal{I}$) is called qs$\mathcal{I}$-*s*-connected if A is not the union of two nonempty qs$\mathcal{I}$-separated sets in (X, $\tau_1, \tau_2, \mathcal{I}$).

Theorem 2.4. Let Y be a biopen subset of an ideal bitopological space (X, $\tau_1, \tau_2, \mathcal{I}$). The following are equivalent:

i. Y is qs$\mathcal{I}$-*s*-connected in (X, $\tau_1, \tau_2, \mathcal{I}$)

ii. Y is qs$\mathcal{I}$- connected in (X, $\tau_1, \tau_2, \mathcal{I}$).

Proof: i) ⇒ ii) Let Y be qs$\mathcal{I}$-*s*-connected in (X, $\tau_1, \tau_2, \mathcal{I}$) and suppose that Y is not qs$\mathcal{I}$-connected in (X, $\tau_1, \tau_2, \mathcal{I}$). There exist non empty disjoint quasi open set A, in Y and qs$\mathcal{I}$- open set B in Y s.t Y = A ∪ B. Since Y is biopen in X and A and B are quasi open and qs$\mathcal{I}$- open in X respectively and A and B are disjoint, then qCl(A) ∩ B = ∅ = A ∩ qs$\mathcal{I}$Cl(B). This implies that A, B are qs$\mathcal{I}$-separated sets in X. Thus, Y is not qs$\mathcal{I}$-*s*-connected in (X, $\tau_1, \tau_2, \mathcal{I}$). Hence we arrive at a contradiction and Y is qs$\mathcal{I}$- connected in (X, $\tau_1, \tau_2, \mathcal{I}$).
ii) ⇒ i) Suppose Y is qs$\mathcal{I}$-connected in (X, $\tau_1, \tau_2, \mathcal{I}$) and Y is not qs$\mathcal{I}$-*s*-connected in (X, $\tau_1, \tau_2, \mathcal{I}$). There exist two qs$\mathcal{I}$-separated sets A, B s.t Y = A∪ B. By Theorem 2.1, A and B are qs$\mathcal{I}$-open and quasi open in Y respectively. Since Y is biopen in X, obviously A and B are qs$\mathcal{I}$- open and quasi open in X respectively. Also Y is qs$\mathcal{I}$-connected so Y cannot be written as the disjoint union of a nonempty quasi open set and a nonempty qs$\mathcal{I}$- open set**.** This is a contradiction and Y is qs$\mathcal{I}$-*s*-connected.

Theorem 2.5. Let (X, τ_1, τ_2, $\mathcal{I}$) be an ideal bitopological space. If A is a qs$\mathcal{I}$-s-connected set of X and H, G are qs$\mathcal{I}$-separated sets of X with A ⊂ H ∪ G, then either A ⊂ H or A ⊂ G.

Proof: Let A ⊂ H ∪ G. Since A = (A ∩ H) ∪ (A ∩ G), then (A ∩ G) ∩ qCl(A ∩ H) ⊂ G ∩ qs$\mathcal{I}$Cl(H) = ∅. By similar reasoning, we have (A ∩ H) ∩ qCl(A ∩ G) ⊂ H ∩ qs$\mathcal{I}$Cl(G) = ∅. If A ∩ H and A ∩ G are nonempty, then A is not qs$\mathcal{I}$- *s*-connected. This is a contradiction. Thus, either A ∩ H = ∅ or A ∩ G = ∅ This implies that either A ⊂ H or A ⊂ G.

Theorem 2.6. If A is a qs$\mathcal{I}$-s-connected set of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) and A ⊂ B ⊂ qCl (A) ∩ qs$\mathcal{I}$Cl(B) then B is qs$\mathcal{I}$-s-connected.

Proof: The theorem can easily be proved by taking the contradiction.

Theorem 2.7. If {M_i: i ∈ I} is a nonempty family of qs$\mathcal{I}$-s-connected sets of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) with $\bigcap_{i\in I}$ Mi ≠ ϕ Then $\bigcup_{i\in I}$ Mi is qs$\mathcal{I}$-s-connected.

Proof: Suppose that $\bigcup_{i\in I}$ Mi is not qs$\mathcal{I}$-s-connected. Then we have $\bigcup_{i\in I}$ Mi = H ∪ G, where H and G are qs$\mathcal{I}$-separated sets in X. Since $\bigcap_{i\in I}$ Mi ≠ ϕ we have a point x in $\bigcap_{i\in I}$ Mi . Since x ∈ $\bigcup_{i\in I}$Mi, either x ε H or x ε G. Suppose that x ε H. Since x ε M_i for each i ε I, then M_i and H intersect for each i ε I. By theorem 2.5: M_i ⊂ H or M_i ⊂ G. Since H and G are disjoint, M_i ⊂ H for all i ε I and hence $\bigcup_{i\in I}$Mi ⊂ H. This implies that G is empty. This is a contradiction. Suppose that x ε G. By similar way, we have that H is empty which is a contradiction. Thus, $\bigcup_{i\in I}$Mi is qs$\mathcal{I}$-s-connected.

Theorem 2.8. Suppose that {M_n: n ε N} is an infinite sequence of qs$\mathcal{I}$-connected open sets of an ideal space (X, τ_1, τ_2, $\mathcal{I}$) and M_n ∩ M_{n+1} ≠ ϕ for each n ε N. Then $\bigcup_{i\in I}$ Mi is qs$\mathcal{I}$-s-connected.

Proof: By induction and Theorems 2.4 and 2.7, the set P_n = $\bigcup_{k\leq n}$Mk is a qs$\mathcal{I}$-connected open set for each n ε N. Also, P_n has a nonempty intersection. Thus $\bigcup_{n\in N}$ Mn is qs$\mathcal{I}$-connected.

Definition 2.4. Let X be an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) and x ∈ X. The union of all qs$\mathcal{I}$-s-connected subsets of X containing x is called the qs$\mathcal{I}$-component of X containing x.

Theorem 2.9. Each qs$\mathcal{I}$-component of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) is a maximal qs$\mathcal{I}$-s connected set of X.

Proof: Obvious.

Theorem 2.10. The set of all distinct qs$\mathcal{I}$-components of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) forms a partition of X.

Proof: Let A and B be two distinct qs$\mathcal{I}$-components of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) containing x and y respectively {x ≠ y}. Suppose that A and B intersect. Then, by Theorem 2.7, A ∪ B is qs$\mathcal{I}$-s-connected in X. Also, A, B ⊆ A ∪ B, so A, B are not maximal and thus A, B are disjoint. Hence they partition X. by induction it can easily be proved that the set of all distinct qs$\mathcal{I}$-components of X forms a partition of X.

Theorem 2.11. Each qs$\mathcal{I}$- component of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) is qs$\mathcal{I}$-closed in X.

Proof: Let A be a qs$\mathcal{I}$-component of X. Therefore qsCl(A) is qs$\mathcal{I}$-s-connected and A = qsCl (A). Thus, A is qs$\mathcal{I}$-closed in X.

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