The Generalized p - k Wright Function And Its Properties

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Abstract

In this paper, we introduce Generalized p-k Wright Function ${}_r^p\psi_s^k$ and discuss about its convergence condition. We obtain functional relation between Generalized p-k Wright function and generalized Wright function and some special cases have also been discussed.

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1 Introduction

The two parameter pochhammer symbol is recently introduce by [8,9], equation 2.1, in the form,

1.1 Definition

Let $x \in C; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Pochhammer Symbol (i.e. Two Parameter Pochhammer Symbol), ${}_p(x)_{n,k}$ is given by

$$ {}_p(x)_{n,k} = (\frac{xp}{k})(\frac{xp}{k} + p)(\frac{xp}{k} + 2p).........(\frac{xp}{k} + (n-1)p). \qquad (1)$$

And the Two Parameter Gamma Function is given by [6],

1.2 Definition

For $x \in C/kZ^-; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Gamma Function (i.e. Two Parameter Gamma Function), ${}_p\Gamma_k(x)$ as,

$$ {}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n!\, p^{n+1} (np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}}. \qquad (2)$$

or

$$ {}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n!\, p^{n+1} (np)^{\frac{x}{k}-1}}{{}_p(x)_{n,k}}. \qquad (3)$$

The integral representation of p - k Gamma Function is given by,

$$ {}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}} t^{x-1} dt. \qquad (4)$$

It is easy to prove following results,

$$ {}_p\Gamma_k(x) = (\frac{p}{k})^{\frac{x}{k}} \Gamma_k(x) = \frac{p^{\frac{x}{k}}}{k} \Gamma(\frac{x}{k}). \tag{5}$$

$$ {}_p(x)_{n,k} = (\frac{p}{k})^n (x)_{n,k} = (p)^n (\frac{x}{k})_n. \tag{6}$$

$$ {}_p(x)_{n,k} = \frac{{}_p\Gamma_k(x+nk)}{{}_p\Gamma_k(x)}. \tag{7}$$

$$ {}_p\Gamma_k(x+k) = \frac{xp}{k} {}_p\Gamma_k(x). \tag{8}$$

$$ {}_p\Gamma_k(x)\, {}_p\Gamma_k(-x) = \frac{\pi}{xk} \frac{1}{sin(\frac{\pi x}{k})}. \tag{9}$$

$$ {}_p\Gamma_k(x)\, {}_p\Gamma_k(k-x) = \frac{p}{k^2} \frac{\pi}{sin(\frac{\pi x}{k})}. \tag{10}$$

$$ n\ {}_p(x)_{n-1,k} = {}_p(x)_{n,k} - {}_p(x-k)_{n,k}. \tag{11}$$

$$ {}_p(x)_{n+j,k} = {}_p(x)_{j,k} \times {}_p(x+jk)_{n,k}. \tag{12}$$

2 Generalized p-k Wright Function

Generalized p-k Wright function is defined as ${}^p_r\psi^k_s$ for $p,k \in R^+ - \{0\}$; $z \in C$, $\alpha_i, \beta_j \in R(\alpha_i, \beta_j \neq 0; i=1,2,...,r; j=1,2,...,s)$ and $(a_i+\alpha_i n),(b_j+\beta_j n) \in C \setminus kZ^-$,

$$ {}^p_r\psi^k_s(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n)} \frac{z^n}{n!}. \tag{13}$$

We use following notations for describing convergence condition,

$$ \Delta = \sum_{j=1}^{s}(\frac{\beta_j}{k}) - \sum_{i=1}^{r}(\frac{\alpha_i}{k}); \quad \delta = \prod_{i=1}^{r} |\frac{\alpha_i}{k}|^{-\frac{\alpha_i}{k}} \prod_{j=1}^{s} |\frac{\beta_j}{k}|^{\frac{\beta_j}{k}}; $$

$$ \mu = \sum_{j=1}^{s}(\frac{b_j}{k}) - \sum_{i=1}^{r}(\frac{a_i}{k}) + \frac{r-s}{2}. $$

Theorem 1 For $p,k \in R^+ - \{0\}$; $z \in C, \alpha_i, \beta_j \in R(\alpha_i, \beta_j \neq 0; i=1,2,...,r; j=1,2,...,s)$

and $(a_i+\alpha_i n),(b_j+\beta_j n)\in C\setminus kZ^-$,

(a) If $\Delta > -1$ then series (2.1) is absolutely convergent for all $z\in C$ and Generalized p-k Wright function ${}_r^p\psi_s^k(z)$ is an entire function of z.

(b) If $\Delta = -1$ then series (2.1) is absolutely convergent for all $|z|<\delta$

and of $|z|=\delta$, $Re(\mu) > \frac{1}{2}$.

Proof Above theorem can prove easily by using results of Diaz and Pariguan [1], Kilbas [10], K.S. Gehlot [5,6].

3 Special cases

For some particular values of the parameters, we can obtain certain Wright function and Mittag-Leffler function defined earlier.

(i) For $p=k$ equation (2.1), reduces in generalized k-Wright Function defined by [5,6].

$$ {}_r^p\psi_s^k(z)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}{}_k\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{s}{}_k\Gamma_k(b_j+\beta_j n)}\frac{z^n}{n!}={}_r\psi_s^k(z). $$

(ii) For $p=k$ and $k=1$ equation (2.1), reduces in generalized Wright Function defined by [10].

$$ {}_r^1\psi_s^1(z)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}{}_1\Gamma_1(a_i+\alpha_i n)}{\prod_{j=1}^{s}{}_1\Gamma_1(b_j+\beta_j n)}\frac{z^n}{n!}={}_r\psi_s(z). $$

(iii) Put $r=1$, $s=2$; $a_1=\gamma$, $\alpha_1=qk$ where $q\in(0,1)\cup N$; $b_1=\beta$, $\beta_1=\alpha$; $b_2=\gamma$, $\beta_2=0$, in equation (2.1) it reduces in p-k Mittag-Leffler Function defined by[8].

$$ {}_1^p\psi_2^k(z)=\sum_{n=0}^{\infty}\frac{{}_p\Gamma_k(\gamma+qkn)}{{}_p\Gamma_k(\beta+\alpha n)\,{}_p\Gamma_k(\gamma)}\frac{z^n}{n!}={}_pE_{k,\alpha,\beta}^{\gamma,q}(z). $$

(iv) Put $r=1$, $s=2$; and $p=k$, $a_1=\gamma$, $\alpha_1=qk$ where $q\in(0,1)\cup N$; $b_1=\beta$, $\beta_1=\alpha$; $b_2=\gamma$, $\beta_2=0$, in equation (2.1) it reduces in generalized k- Mittag-Leffler Function defined by[3].

$$ {}_1^k\psi_2^k(z) = \sum_{n=0}^{\infty} \frac{{}_k\Gamma_k(\gamma+qkn)}{{}_k\Gamma_k(\beta+\alpha n)\,{}_k\Gamma_k(\gamma)} \frac{z^n}{n!} = GE_{k,\alpha,\beta}^{\gamma,q}(z). $$

(v) Put $r=1$, $s=2$ and $p=k$, $a_1=\gamma$, $\alpha_1=k$; $b_1=\beta$, $\beta_1=\alpha$; $b_2=\gamma$, $\beta_2=0$, in equation (2.1) it reduces in k-Mittag-Leffler Function defined by[2].

$$ {}_1^k\psi_2^k(z) = \sum_{n=0}^{\infty} \frac{{}_k\Gamma_k(\gamma+kn)}{{}_k\Gamma_k(\beta+\alpha n)\,{}_k\Gamma_k(\gamma)} \frac{z^n}{n!} = E_{k,\alpha,\beta}^{\gamma}(z). $$

(vi) Put $r=1$, $s=2$; and $p=k=1$, $a_1=\gamma$, $\alpha_1=1$; $b_1=\beta$, $\beta_1=\alpha$; $b_2=\gamma$, $\beta_2=0$, in equation (2.1) it reduces in Mittag-Leffler Function defined by[12].

$$ {}_1^1\psi_2^1(z) = \sum_{n=0}^{\infty} \frac{{}_1\Gamma_1(\gamma+n)}{{}_1\Gamma_1(\beta+\alpha n)\,{}_1\Gamma_1(\gamma)} \frac{z^n}{n!} = E_{\alpha,\beta}^{\gamma}(z). $$

(vii) Put $r=1$, $s=2$ and $p=k=1$, $a_1=1$, $\alpha_1=1$; $b_1=\beta$, $\beta_1=\alpha$; $b_2=1$, $\beta_2=0$, in equation (2.1) it reduces in Mittag-Leffler Function defined by[13].

$$ {}_1^1\psi_2^1(z) = \sum_{n=0}^{\infty} \frac{\Gamma_1(1+n)}{\Gamma_1(\beta+\alpha n)\Gamma_1(1)} \frac{z^n}{n!} = E_{\alpha,\beta}(z). $$

(viii) Put $r=1$, $s=2$ and $p=k=1$, $a_1=1$, $\alpha_1=1$; $b_1=1$, $\beta_1=\alpha$; $b_2=1$, $\beta_2=0$, in equation (2.1) it reduces in Mittag-Leffler Function defined by[11].

$$ {}_1^1\psi_2^1(z) = \sum_{n=0}^{\infty} \frac{\Gamma_1(1+n)}{\Gamma_1(1+\alpha n)\Gamma_1(1)} \frac{z^n}{n!} = E_{\alpha}(z). $$

4 Properties of Generalized p-k Wright Function

We evaluate the functional relation between Generalized p-k Wright Function and Generalized Wright Function and we evaluate the recurrence relations of Generalized p-k Wright Function.

Theorem 2 The functional relation between Generalized p-k Wright function and generalized Wright function, is given by,

$$ {}_{r}^{p}\psi_{s}^{k}\left[\begin{matrix}(a_i,\alpha_i)_{1,r};\\ \\(b_j,\beta_j)_{1,s};\end{matrix}\; z\right]=\frac{p^{\sum_{i=1}^{r}(\frac{a_i}{k})-\sum_{j=1}^{s}(\frac{b_j}{k})}}{k^{r-s}}\;{}_{r}\psi_{s}\left[\begin{matrix}(\frac{a_i}{k},\frac{\alpha_i}{k})_{1,r};\\ \\(\frac{b_j}{k},\frac{\beta_j}{k})_{1,s};\end{matrix}\; zp^{\sum_{i=1}^{r}(\frac{\alpha_i}{k})-\sum_{j=1}^{s}(\frac{\beta_j}{k})}\right]. \tag{14} $$

Or the counter part,

$$ {}_{r}\psi_{s}\left[\begin{matrix}(a_i,\alpha_i)_{1,r};\\ \\(b_j,\beta_j)_{1,s};\end{matrix}\; z\right]=\frac{p^{\sum_{j=1}^{s}(b_j)-\sum_{i=1}^{r}(a_i)}}{k^{s-r}}\;{}_{r}^{p}\psi_{s}^{k}\left[\begin{matrix}(ka_i,k\alpha_i)_{1,r};\\ \\(kb_j,k\beta_j)_{1,s};\end{matrix}\; zp^{\sum_{j=1}^{s}(\beta_j)-\sum_{i=1}^{r}(\alpha_i)}\right]. \tag{15} $$

Proof Consider the right hand side of (4.1), and using equation (2.1), we have,

$$ A\equiv {}_{r}^{p}\psi_{s}^{k}\left[\begin{matrix}(a_i,\alpha_i)_{1,r};\\(b_j,\beta_j)_{1,s};\end{matrix}\; z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}{}_{p}\Gamma_{k}(a_i+\alpha_i n)}{\prod_{j=1}^{s}{}_{p}\Gamma_{k}(b_j+\beta_j n)}\frac{z^n}{n!}. $$

Using equation (1.5), we have,

$$ A\equiv\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\frac{p^{\frac{a_i+\alpha_i n}{k}}}{k}\Gamma(\frac{(a_i+\alpha_i n)}{k})}{\prod_{j=1}^{s}\frac{p^{\frac{b_j+\beta_j n}{k}}}{k}\Gamma(\frac{(b_j+\beta_j n)}{k})}\frac{z^n}{n!} $$

$$ A\equiv\frac{p^{\frac{\sum_{i=1}^{r}a_i-\sum_{j=1}^{s}b_j}{k}}}{k^{r-s}}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\Gamma(\frac{(a_i+\alpha_i n)}{k})}{\prod_{j=1}^{s}\Gamma(\frac{(b_j+\beta_j n)}{k})}\frac{(zp^{\frac{\sum_{i=1}^{r}\alpha_i-\sum_{j=1}^{s}\beta_j}{k}})^n}{n!} $$

$$A \equiv \frac{p^{\sum_{i=1}^{r}(\frac{a_i}{k})-\sum_{j=1}^{s}(\frac{b_j}{k})}}{k^{r-s}} \, {}_r\psi_s \left[\begin{matrix} (\frac{a_i}{k},\frac{\alpha_i}{k})_{1,r}; \\ \qquad zp^{\sum_{i=1}^{r}(\frac{\alpha_i}{k})-\sum_{j=1}^{s}(\frac{\beta_j}{k})} \\ (\frac{b_j}{k},\frac{\beta_j}{k})_{1,s}; \end{matrix} \right].$$

Hence.

Similarly we can Prove counterpart, (4.2).

Theorem 3 Let $p,k \in R^{+}-\{0\};\ z \in C,\ \alpha_i,\beta_j \in R(\alpha_i,\beta_j \neq 0; i=1,2,...,r; j=1,2,...,s)$ and $(a_i+\alpha_i n),(b_j+\beta_j n) \in C \setminus kZ^{-}$, then,

$${}_r^p\psi_{s+1}^k \left[\begin{matrix} (a_i,\alpha_i)_{1,r}; \\ z \\ (b_j,\beta_j)_{1,s},(b,\beta); \end{matrix} \right] = \frac{bp}{k} \, {}_r^p\psi_{s+1}^k \left[\begin{matrix} (a_i,\alpha_i)_{1,r}; \\ z \\ (b_j,\beta_j)_{1,s},(b+k,\beta); \end{matrix} \right]$$

$$+\frac{\beta zp}{k}\frac{d}{dz} {}_r^p\psi_{s+1}^k \left[\begin{matrix} (a_i,\alpha_i)_{1,r}; \\ z \\ (b_j,\beta_j)_{1,s},(b+k,\beta); \end{matrix} \right]. \qquad (16)$$

Proof Consider the right hand side of (4.3) and using the definition of Generalized p-k Wright function (2.1), we have,

$$B \equiv \frac{pb}{k}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n)\,{}_p\Gamma_k(b+k+\beta n)}\frac{z^n}{n!}$$

$$+\frac{\beta zp}{k}\frac{d}{dz}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n)\,{}_p\Gamma_k(b+k+\beta n)}\frac{z^n}{n!}.$$

Using equation (1.8), we get,

$$B \equiv \sum_{n=0}^{\infty}\frac{p}{k}\frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i+\alpha_i n)(b+\beta n)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n)\,{}_p\Gamma_k(b+k+\beta n)}\frac{z^n}{n!}$$

$$B \equiv \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n)\,{}_p\Gamma_k(b+\beta n)}\frac{z^n}{n!}.$$

this immediately leads to,

$$ {}_{r}^{p}\psi_{s+1}^{k}\left[\begin{matrix}(a_i,\alpha_i)_{1,r}; \\ & z \\ (b_j,\beta_j)_{1,s},(b,\beta);\end{matrix}\right]. $$

Hence.

Theorem 4 Let $p,k \in R^{+}-\{0\};\ z\in C, \alpha_i,\beta_j \in R(\alpha_i,\beta_j \neq 0; i=1,2,...,r; j=1,2,...,s)$ and $(a_i+\alpha_i n),(b_j+\beta_j n)\in C\setminus kZ^{-}$, then,

$$ {}_{r+1}^{p}\psi_{s+1}^{k}\left[\begin{matrix}(a_i,\alpha_i)_{1,r},(a+k,\alpha k); \\ & z \\ (b_j,\beta_j)_{1,s},(a+k,0);\end{matrix}\right] $$

$$ -\ {}_{r+1}^{p}\psi_{s+1}^{k}\left[\begin{matrix}(a_i,\alpha_i)_{1,r},(a,\alpha k); \\ & z \\ (b_j,\beta_j)_{1,s},(a,0);\end{matrix}\right] $$

$$ = z\alpha\ {}_{p}(\alpha+k)_{\alpha-1,k}\ {}_{r+1}^{p}\psi_{s+1}^{k}\left[\begin{matrix}(a_i,\alpha_i)_{1,r},(a+\alpha k,\alpha k); \\ & z \\ (b_j+\beta_j,\beta_j)_{1,s},(a+\alpha k,0);\end{matrix}\right]. \tag{17} $$

Proof Consider the right hand side of (4.4) and using the definition of Generalized p-k Wright function (2.1),
we have,

$$ D \equiv {}_{r+1}^{p}\psi_{s+1}^{k}\left[\begin{matrix}(a_i,\alpha_i)_{1,r},(a+k,\alpha k); \\ & z \\ (b_j,\beta_j)_{1,s},(a+k,0);\end{matrix}\right] $$

$$ -\ {}_{r+1}^{p}\psi_{s+1}^{k}\left[\begin{matrix}(a_i,\alpha_i)_{1,r},(a,\alpha k); \\ & z \\ (b_j,\beta_j)_{1,s},(a,0);\end{matrix}\right] $$

$$ D \equiv \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}{}_{p}\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{s}{}_{p}\Gamma_k(b_j+\beta_j n)}\frac{z^n}{n!} $$

$$ \left(\frac{{}_{p}\Gamma_k(a+k+\alpha kn)}{{}_{p}\Gamma_k(a+k)}-\frac{{}_{p}\Gamma_k(a+\alpha kn)}{{}_{p}\Gamma\Gamma_k(a)}\right). $$

Using equation (1.11) and (1.12), we get,

$$D \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n)} \frac{z^n}{n!} \left({}_p(a+k)_{\alpha n,k} - {}_p(a)_{\alpha n,k} \right)$$

$$D \equiv \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n)} \frac{z^n}{n!} \left(n\alpha \ {}_p(a+k)_{\alpha n-1,k} \right)$$

$$D \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i+\alpha_i (n+1))}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j (n+1))} \frac{z^{n+1}}{n+1!} \left(n\alpha \ {}_p(a+k)_{\alpha n+\alpha-1,k} \right.$$

$$D \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i+\alpha_i n+\alpha_i)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n+\beta_j)} \frac{z^{n+1}}{n+1!}$$

$$\left({}_p(a+k)_{\alpha-1,k} \ {}_p(a+k+(\alpha-1)k)_{n\alpha,k} \right)$$

$$D \equiv \alpha z \ {}_p(a+k)_{\alpha-1,k} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i+\alpha_i n+\alpha_i)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n+\beta_j)}$$

$$\frac{z^n(n+1)}{n+1!} \left({}_p(a+\alpha k)_{\alpha n,k} \right).$$

Using equation (2.1), we have,

$$= z\alpha \ {}_p(\alpha+k)_{\alpha-1,k} \ {}_{r+1}^{\ p}\psi_{s+1}^{k} \left[\begin{matrix} (a_i,\alpha_i)_{1,r}, (a+\alpha k, \alpha k); \\ \quad z \\ (b_j+\beta_j, \beta_j)_{1,s}, (a+\alpha k, 0); \end{matrix} \right].$$

Hence Proved.

5 Integral Representation of Generalized p-k Wright Function

Theorem 5 Let $p,k \in R^+ - \{0\}$; $z \in C, \alpha_i, \beta_j \in R(\alpha_i, \beta_j \neq 0; i = 1,2,...,r; j = 1,2,...,s)$ and $(a_i+\alpha_i n), (b_j+\beta_j n) \in C \setminus kZ^-$, then,

$$ {}_r^p\psi_s^k\left[\begin{matrix}(a_i,k\alpha_i)_{1,r}; & \\ & z\\ (b_j,k\beta_j)_{1,s}; & \end{matrix}\right] $$

$$ =\frac{\prod_{i=1}^{r}{}_p\Gamma_k(a_i)}{\prod_{j=1}^{s}{}_p\Gamma_k(b_j)}\prod_{l=1}^{\alpha_i}\prod_{m=1}^{\beta_j}\frac{1}{B(\mu_l,\nu_m-\mu_l)}\int_0^1 t^{\mu_l-1}(1-t)^{\nu_m-\mu_l-1}e^{\frac{(\alpha_i p)^{\alpha_i}tz}{(p\beta_j)^{\beta_j}}}dt. \tag{18} $$

Proof Using the definition of Generalized p-k Wright function

$$ {}_r^p\psi_s^k\left[\begin{matrix}(a_i,k\alpha_i)_{1,r}; & \\ & z\\ (b_j,k\beta_j)_{1,s}; & \end{matrix}\right] $$

$$ =\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}{}_p\Gamma_k(a_i+k\alpha_i n)}{\prod_{j=1}^{s}{}_p\Gamma_k(b_j+k\beta_j n)}\frac{z^n}{n!} $$

$$ =\sum_{n=0}^{\infty}D\frac{z^n}{n!}. \tag{19} $$

Where,

$$ D\equiv\frac{\prod_{i=1}^{r}{}_p\Gamma_k(a_i+k\alpha_i n)}{\prod_{j=1}^{s}{}_p\Gamma_k(b_j+k\beta_j n)} $$

$$ \equiv\frac{\prod_{i=1}^{r}{}_p\Gamma_k(a_i)\,{}_p(a_i)_{\alpha_i n,k}}{\prod_{j=1}^{s}{}_p\Gamma_k(b_j)\,{}_p(b_j)_{\beta_j n,k}}. $$

Using equation (1.7),

$$ \equiv\frac{\prod_{i=1}^{r}{}_p\Gamma_k(a_i)(p)^{\alpha_i n}(\frac{a_i}{k})_{\alpha_i n}}{\prod_{j=1}^{s}{}_p\Gamma_k(b_j)(p)^{\beta_j n}(\frac{b_j}{k})_{\beta_j n}}. $$

Using, $(\gamma)_{nq} = q^{nq} \prod_{l=1}^{q} (\frac{\gamma+l-1}{q})_n$ if $q\in N$ then, we have,

$$D \equiv \frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i)(p)^{\alpha_i n}(\alpha_i)^{\alpha_i n}}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j)(p)^{\beta_j n}(\beta_j)^{\beta_j n}} \frac{\prod_{l=1}^{\alpha_i}\left(\frac{\frac{\alpha_i}{k}+l-1}{\alpha_i}\right)_n}{\prod_{m=1}^{\beta_j}\left(\frac{\frac{\beta_j}{k}+m-1}{\beta_j}\right)_n}.$$

Let, $\frac{\frac{\alpha_i}{k}+l-1}{\alpha_i} = \mu_l$ and $\frac{\frac{\beta_j}{k}+m-1}{\beta_j} = \nu_m$ then,

$$D \equiv \frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i)(p)^{\alpha_i n}(\alpha_i)^{\alpha_i n}}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j)(p)^{\beta_j n}(\beta_j)^{\beta_j n}} \prod_{l=1}^{\alpha_i} \prod_{m=1}^{\beta_j} \frac{\Gamma \nu_m}{\Gamma(\nu_m-\mu_l)\,\Gamma(\mu_l)} \times \frac{\Gamma(\mu_l+n)\,\Gamma(\nu_m-\mu_l)}{\Gamma(\nu_m-\mu_l+n)}.$$

above equation can be written in the form of beta function as,

$$D \equiv \frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j)} \frac{p^{\alpha_i n}}{p^{\beta_j n}} \frac{(\alpha_i^{\alpha_i n})}{(\beta_j)^{\beta_j n}} \prod_{l=1}^{\alpha_i} \prod_{m=1}^{\beta_j} \frac{\Gamma \nu_m}{\Gamma(\nu_m-\mu_l)\,\Gamma(\mu_l)} \int_0^1 t^{\mu_l+n-1}(1-t)^{\nu_m-\mu_l-1}dt.$$

Using in equation (5.2), we have,

$$ {}_r^p\psi_s^k \left[\begin{matrix} (a_i, k\alpha_i)_{1,r}; \\ \\ (b_j, k\beta_j)_{1,s}; \end{matrix} \; z \right]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j)} \frac{p^{\alpha_i n}}{p^{\beta_j n}} \frac{(\alpha_i^{\alpha_i n})}{(\beta_j)^{\beta_j n}} \prod_{l=1}^{\alpha_i} \prod_{m=1}^{\beta_j} \frac{\Gamma \nu_m}{\Gamma(\nu_m-\mu_l)\,\Gamma(\mu_l)} \frac{z^n}{n!} \int_0^1 t^{\mu_l+n-1}(1-t)^{\nu_m-\mu_l-1}dt.$$

Using definition of beta function, we have

$$= \frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j)} \prod_{l=1}^{\alpha_i} \prod_{m=1}^{\beta_j} \frac{1}{B(\mu_l, \nu_m-\mu_l)} \int_0^1 t^{\mu_l-1}(1-t)^{\nu_m-\mu_l-1} e^{\frac{(\alpha_i p)^{\alpha_i} tz}{(p\beta_j)^{\beta_j}}} dt.$$

Hence Proved.

Theorem 6 Let $p,k \in R^+ - \{0\}$; $z \in C, \alpha_i, \beta_j \in R(\alpha_i, \beta_j \neq 0; i = 1,2,...,r; j = 1,2,...,s)$ and $(a_i + \alpha_i n), (b_j + \beta_j n) \in C \setminus kZ^-$, then,

$$ {}_r^p\psi_s^k\left[\begin{array}{c} (a_i,\alpha_i)_{1,r}; \\ \quad z \\ (b_j,\beta_j)_{1,s}; \end{array}\right] $$

$$ = \frac{p^{\sum_{i=1}^{r}(\frac{a_i}{k})}}{k^r}\int_0^\infty e^{-t}t^{\sum_{i=1}^{r}(\frac{a_i}{k})-r}\ {}_0^p\psi_s^k\left[\begin{array}{c} - \\ z(pt)^{\sum_{i=1}^{r}(\frac{\alpha_i}{k})} \\ (b_j,\beta_j)_{1,s} \end{array}\right]. \qquad (20) $$

Proof Using the definition of Generalized p-k Wright function, we have,

$$ {}_r^p\psi_s^k\left[\begin{array}{c} (a_i,\alpha_i)_{1,r}; \\ \quad z \\ (b_j,\beta_j)_{1,s}; \end{array}\right] $$

$$ = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r} {}_p\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n)}\frac{z^n}{n!} $$

$$ = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\frac{p^{\frac{a_i+\alpha_i n}{k}}}{k}\Gamma(\frac{a_i+\alpha_i n}{k})}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n)}\frac{z^n}{n!} $$

$$ = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\frac{p^{\frac{a_i+\alpha_i n}{k}}}{k}}{\prod_{j=1}^{s} {}_p\Gamma_k(b_j+\beta_j n)n!}\int_0^\infty e^{-t}t^{\frac{a_i+\alpha_i n}{k}-1}dt $$

$$ = \frac{p^{\sum_{i=1}^{r}(\frac{a_i}{k})}}{k^r}\int_0^\infty e^{-t}t^{\sum_{i=1}^{r}(\frac{a_i}{k})-r}\ {}_0^p\psi_s^k\left[\begin{array}{c} -; \\ z(pt)^{\sum_{i=1}^{r}(\frac{\alpha_i}{k})} \\ (b_j,\beta_j)_{1,s}; \end{array}\right] $$

. Hence Proved.

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