NEW SOLUTIONS FOR SOME HIGHER-ORDER DIFFERENTIAL EQUATIONS

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I**Abstract :-**
In this article, we apply two methods to find the exact solutions for some nonlinear evolution equations in mathematical physics, the homogenous balance method and the modified kudryashov method With the aid of computer algebraic system Mathematica. We apply the modified kudryashov method to construct the exact traveling wave solutions of the nonlinear fifth-order Sawada-Kotera equation and the nonlinear fifth-lax equation.
We apply the homogeneous balance method to construct the exact traveling wave solutions of the nonlinear seventh-order Sawada-Kotera equation and the nonlinear seventh-lax equation. These equations have wide applications in quantum mechanics and nonlinear optics.

Keywords: the homogenous balance method, the modified kudryashov method, nonlinear evolution equations, exact solutions.

1.Introduction

There is no doubt that finding solutions to partial differential equations has an important role in the interpretation of many physical phenomena, especially fluid mechanics and other branches of physics such as sound, heat, static electricity, fluid flow and flexibility.
An important equation in the interpretation of these phenomena is the nonlinear partial differential equations. Many methods have been used to find solutions to these equations and we have proven their strength, such as scattering transformation[2], the Hirota method [3], the Exp-function method[3], the simplest equation method[4], the Tanh-fuction method, the sine-cosine method[5],the Variational Iteration method [6], the Adomain decomposition method[5] and numerical approximation of the differential equations which are enabling us to find the approximate solutions of the differential equations. Several numerical techniques such as finite element, Euler, Runga-Kutta, etc. are used for finding the approximate solution of these types of differential equations[17].

The objective of this paper is to employ two methods the modified kudryashov[1] method for finding the exact soliton solutions of the nonlinear fifth-order Sawada-Kotera equation and the nonlinear fifth-order lax equation in terms of symmetrical sine and Lucas cosine functions and the homogenous balance[8] method for finding the exact soliton solutions of the the nonlinear seventh-order Sawada-Kotera equation and the nonlinear seventh -order lax equation, these equations have been discussed before be many authors using other methods such as the Hirota direct method[3] , the Tanh-coth method and the Sech method[5]. This paper is organized as follows : In section 2, we give the description of the modified kudryashov method. In section 3, we apply this method with the aid of Mathematica to solve two fifth-order nonlinear PDEs indicated above. In section 4, we give the description of the homogeneous balance method. In section 5, we apply this method to solve two seventh-order nonlinear PDEs indicated above. In section 6, we present the physical explanation of the obtained soliton. In section 7, some conclusions are given. .

2. Description of the modified kudryashov method

suppose we have a nonlinear evolution equation in the form

$$F(u_x, u_t, u_{tx}, u_{xx}, \ldots\ldots) = 0 \tag{2.1}$$

Where F is a polynomial in u(x , t) and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [9],[11],[13]::

Step .1

Using wave transformation

$$\boldsymbol{u(x,t) = u(\zeta), \quad \zeta = k\,x + \omega t} \tag{2.2}$$

to reduce equation (2.1) to the following ODE :

$$\boldsymbol{F(u', u', u'', \ldots\ldots) = 0} \tag{2.3}$$

Where F is a polynomial in u(ζ) and its total derivatives, while k, ω are constants and

$$' = \frac{\boldsymbol{du}}{\boldsymbol{d\zeta}} \tag{2.4}$$

Step.2

We suppose that equation (2.3) has the formal solution

$$u(\zeta) = \sum_{n=i}^{N} a_n\, Q(\zeta)^n \tag{2.4}$$

where a_n (n= 0,1,2,3…….,N) are constants to be determined, such that $a_N \neq 0$, and Q(ζ) is the solution of the equation

$$Q'(\zeta) = [Q(\zeta)2 - Q(\zeta)]\, ln(a) \tag{2.5}$$

Equation (2.5) has the solution

$$Q(\zeta) = \frac{1}{1 \pm a^{\zeta}} \tag{2.6}$$

Step. 3

We determine the positive integer N in equation (2.4) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in equation (2.3)

Step .4

Substitute equation (2.4) in to equation (2.3), we calculate the necessary derivatives u', u'' ,……of the function u(ζ). As a result of this substitution , we get a polynomial of Q^i (i= 0, 1, 2,……….). In this polynomial we gather all terms of same powers and equating them to zero, we obtain a system of algebraic equations which can be solved be Maple or Mathematica to get the unknown parameters a_n(n= 0, 1 ,2 ,….N) , k and ω. Consequently ,we obtain the exact solutions of equation (2.1)

Remark. the obtained solutions can be depended on the symmetrical hyperbolic Lucas functions and Fibonacii functions proposed by Stakhov and Rozin [16]. The symmetrical Lucas sine, cosine, tangent and cotangent functions are respectively, defined as

$$s\,L\,s(\zeta) = a^{\zeta} - a^{\zeta}, c\,L\,s(\zeta) = a^{\zeta} + a^{\zeta} \tag{2.7}$$

$$t\,Ls\,(\zeta) = \frac{a^{\zeta} - a^{\zeta}}{a^{\zeta} + a^{\zeta}} = \frac{s\,L\,s(\zeta)}{c\,L\,s(\zeta)}, ct\,L\,s(\zeta) = \frac{a^{\zeta}+a^{-\zeta}}{a^{\zeta}-a^{-\zeta}} = \frac{c\,L\,s(\zeta)}{s\,L\,s(\zeta)} \tag{2.8}$$

$$s\,F\,s\,(\zeta) = \frac{a^{\zeta}-a^{-\zeta}}{\sqrt{5}}, \quad c\,F\,s\,(\zeta) = \frac{a^{\zeta}+a^{-\zeta}}{\sqrt{5}}, \quad t\,F\,s\,(\zeta) = \frac{a^{\zeta}-a^{-\zeta}}{a^{\zeta}+a^{-\zeta}} \tag{2.9}$$

also these functions satisfy the following formulas :

$$[c\,L\,s\,(\zeta)]^2 - [s\,L\,s\,(\zeta)]^2 = 4 \tag{2.10}$$

$[c\,F\,s\,(\zeta)\,]^2$ - $[s\,F\,s\,(\zeta)\,]^2 = \frac{4}{5}$

The obtained solutions in this paper can be obtained in terms of the symmetrical hyperbolic lucas functions.

3.Illustrative examples.

In this section, we apply the modified kudryashov method to find the exact solutions of the following nonlinear partial differential equations:

Example 1.

The fifth order Sawada-Kotera (Sk) equation

This equation is well known[5],[3], [6],and has the form

$$u_t + 5u^2 u_x + 5u_x u_{xx} + 5u u_{3x} + u_{5x} = 0 \tag{3.1}$$

let us now solve Eq(3.1) by using the modified kudryashov method. To this end we use the wave transformation $\zeta = k\,x + \omega\,t$ to reduce equation (3.1) to the following ODE :

$$\omega u' + 5\,k\,u^2\,u' + 5k^3\,u'\,u'' + 5\,k^3 u\,u''' + k^5 u^{(5)} = 0$$
(3.2)

Balancing $u^{(5)}$ with $u^2 u'$ yields N= 2. Consequently, Eq (3.2) has the formal solution

$$u(\zeta) = a_0 + a_1 Q(\zeta) + a_2\, Q(\zeta)^2$$
(3.3)

where a_0, a_1 and a_2 are constants to be determined such that $a_2 \neq 0$ from (3.3)

, we can obtain

$$u' = (\ln a)\,(a_1 + 2\,a_2 Q)\,(Q-1)\,Q\ . \tag{3.4}$$

$$u'' = (\ln a)[(-1+2Q)\,a_1 + 2Q\,(3Q-2)\,a_2](Q-1)\,2Q\,. \tag{3.5}$$

$$u''' = (\ln a)[(1-6Q+6Q^2)\,a_1 + 2Q\,(4-15Q+12Q^2)a_2]\,(Q-1)\,3Q. \tag{3.6}$$

$$u^{(4)} = (\ln a)[(-1+14Q-36Q^2+24\,Q^3)a_1 + 2Q\,(-8+57Q-108\,Q^2+60\,Q^2)\,a_2](Q-1)\,4Q. \tag{3.7}$$

$$u^{(5)} = (\ln a)\,[(1-30\,Q+150\,Q^2-240\,Q^3+120\,Q^4)\,a_1 + 2Q\,(16-195Q+660\,Q^2-840\,Q^3+360\,Q^4)\,a_2](Q-1)\,5\,Q. \tag{3.8}$$

substituting (3.3) – (3.8) in to (3.2) and equating all the coefficients of powers of Q (ζ) to zero, we obtain some algebraic equations.

On solving the algebraic equations using Mathematica or maple, we get the following results:

Case.1

$$a_0 = \frac{1}{4}\left(-3k^2\text{Ln}[a]^2 - \frac{\sqrt{(-8k+9k^6)\text{Ln}[a]^4}}{k}\right),\quad a_1 = 3\left(3k^2\text{Ln}[a]^2 + \frac{\sqrt{(-8k+9k^6)\text{Ln}[a]^4}}{k}\right)$$

$$a_2 = \frac{3\left(-3k^3\text{Ln}[a]^2 - \sqrt{-8k\text{Ln}[a]^4 + 9k^6\text{Ln}[a]^4}\right)}{k} \tag{3.9}$$

$$\omega = \frac{1}{8}\left(12\text{Ln}[a]^4 - 15k^5\text{Ln}[a]^4 - 5k^2\text{Ln}[a]^2\sqrt{(-8k+9k^6)\text{Ln}[a]^4}\right)$$

From (2.6), (2.7), (3.3) and (3.4) we obtain the following exact solutions of Eq (3.1)

$$u_1(x,t) = \frac{1}{4}\left(-3k^2\text{Ln}[a]^2 - \frac{\sqrt{(-8k+9k^6)\text{Ln}[a]^4}}{k}\right) + 3\left(3k^2 + \frac{\sqrt{(-8k+9k^6)}}{k}\right)\left(\frac{\ln[a]}{c\,L\,s\left(\frac{\xi}{2}\right)}\right)^2 \tag{3.10}$$

$$u_2(x,t) = \frac{1}{4}\left(-3k^2\text{Ln}[a]^2 - \frac{\sqrt{(-8k+9k^6)\text{Ln}[a]^4}}{k}\right) - 3\left(3k^2 + \frac{\sqrt{(-8k+9k^6)}}{k}\right)\left(\frac{\ln[a]}{s\,L\,s\left(\frac{\xi}{2}\right)}\right)^2 \tag{3.11}$$

Where

$$\zeta = k\ \mathbf{x} + \left(\frac{1}{8}\left(-3(-4+5k^5)Ln[a]^4 - 5k^2 Log[a]^2\sqrt{k(-8+9k^5)Ln[a]^4}\right)\right)\mathbf{t}$$

Case.2

$$a_0 = \frac{1}{4}(-3k^2\mathrm{Ln}[a]^2 + \frac{\sqrt{(-8k+9k^6)\mathrm{Ln}[a]^4}}{k}), \quad a_1 = 3(3k^2\mathrm{Ln}[a]^2 - \frac{\sqrt{(-8k+9k^6)\mathrm{Ln}[a]^4}}{k})$$

$$a_2 = \frac{3(-3k^3\mathrm{Ln}[a]^2+\sqrt{-8k\mathrm{Ln}[a]^4+9k^6\mathrm{Ln}[a]^4})}{k} \qquad (3.12)$$

$$\omega = \frac{1}{8}(12\mathrm{Ln}[a]^4 - 15k^5\mathrm{Ln}[a]^4 + 5k^2\mathrm{Ln}[a]^2\sqrt{(-8k+9k^6)\mathrm{Ln}[a]^4})$$

In this case, we deduce the following exact solutions of Eq(3.1)

$$u_3(x,t) = \frac{-3k^3\mathrm{Ln}[a]^2 + \sqrt{(-8k+9k^6)\mathrm{Ln}[a]^4}}{4k} - \frac{3(3k^3+\sqrt{-8k+9k^6})}{k}(\frac{\ln[a]}{s\,Ls(\zeta)})^2 \qquad (3.13)$$

$$u_4(x,t) = \frac{-3k^3\mathrm{Ln}[a]^2 + \sqrt{(-8k+9k^6)\mathrm{Ln}[a]^4}}{4k} + \frac{3(3k^3+\sqrt{-8k+9k^6})}{k}(\frac{\ln[a]}{c\,Ls(\zeta)})^2 \qquad (3.14)$$

3. Example 2

The nonlinear fifth order lax equation,

This equation well known[5],[9], and has the form

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{3x} + u_{5x} = 0 \qquad (3.15)$$

let us now solve Eq(3.10) by using the modified kudryashov method . to this end , we use the wave transformation $\zeta = k\,x + \omega\,t$ to reduce Eq (3.11)

$$\omega u' + 30ku^2u' + 20k^3u'u'' + 10k^3u\,u^{(3)} + u^{(5)} = 0 \qquad (3.11)$$

Balancing $u^{(5)}$ with u^2u' yields N=2. Consequently, Eq (3.11) has the formal solution (3.3). substituting (3.3)- (3.8) in (3.11) and equating all the coefficients of powers of Q(ζ) to zero, we obtain some algebraic equations :

On solving these algebraic equations using the Mathematica or Maple, we get the following On solving these algebraic equations using the Mathematica or Maple, we get the following result :

Case.1

$$a_0 = \frac{1}{6}(-2k^2\mathrm{Ln}[a]^2 - \frac{\sqrt{(-3k+4k^6)\mathrm{Ln}[a]^4}}{k}), \; a_1 = \frac{2(2k^3\mathrm{Ln}[a]^2+\sqrt{-3k\mathrm{Ln}[a]^4+4k^6\mathrm{Ln}[a]^4})}{k})$$

$$a_2 = \frac{-2(2k^3\mathrm{Ln}[a]^2+\sqrt{-3k\mathrm{Ln}[a]^4+4k^6\mathrm{Ln}[a]^4})}{k} \qquad (3.12)$$

$$\omega = \frac{1}{6}(9\mathrm{Ln}[a]^4 - 20k^5\mathrm{Ln}[a]^4 - 10k^2\mathrm{Ln}[a]^2\sqrt{(-3k+4k^6)\mathrm{Ln}[a]^4})$$

From (2.6), (2.7), (3.3) and (3.12) we obtain the following exact solutions of Eq.(3.10)

$$u_1(x,t) = \frac{-2k^3\ln(a)^2 - \sqrt{(-3k+4k^6)\mathrm{Ln}[a]^4}}{6\,k} - (\frac{2(2k^3+\sqrt{-3k+4k^6})}{k})\,(\frac{\ln(a)}{s\,L\,s\,(\frac{\xi}{2})})^2 \qquad (3.13)$$

$$u_2(x,t) = \frac{-2k^3\ln(a)^2 - \sqrt{(-3k+4k^6)\mathrm{Ln}[a]^4}}{6\,k} + (\frac{2(2k^3+\sqrt{-3k+4k^6})}{k})\,(\frac{\ln(a)}{c\,L\,s\,(\frac{\xi}{2})})^2 \qquad (3.14)$$

$$where \;\; \zeta = k\,x + \frac{1}{6}\left(9\mathrm{Ln}[a]^4 - 20k^5\mathrm{Ln}[a]^4 - 10k^2\mathrm{Ln}[a]^2\sqrt{(-3k+4k^6)\mathrm{Ln}[a]^4}\right)\; t$$

Case.2

$a_0 = \frac{-2k^3\text{Ln}[a]^2+\sqrt{(-3k+4k^6)\text{Ln}[a]^4}}{6\,k}$, $a_1 = -\frac{2(2k^3\text{Ln}[a]^2+\sqrt{(-3k+4k^6)\text{Ln}[a]^4})}{k}$

$a_2 = \frac{-2(2k^3\text{Ln}[a]^2-\sqrt{-3k\text{Ln}[a]^4+4k^6\text{Ln}[a]^4})}{k}$ (3.15)

$\omega = \frac{1}{6}(9\text{Ln}[a]^4 - 20k^5\text{Ln}[a]^4 + 10k^2\text{Ln}[a]^2\sqrt{(-3k+4k^6)\text{Ln}[a]^4})$

$$u_3(x,t) = \frac{-2k^3\text{Ln}[a]^2+\sqrt{(-3k+4k^6)\text{Ln}[a]^4}}{6\,k} - \frac{2(2k^3+\sqrt{(-3k+4k^6)})}{k}\left(\frac{\ln(a)}{s\,L\,s\,(\frac{\xi}{2})}\right)^2 \quad (3.16)$$

$$u_4(x,t) = \frac{-2k^3\text{Ln}[a]^2+\sqrt{(-3k+4k^6)\text{Ln}[a]^4}}{6\,k} + \frac{2(2k^3+\sqrt{(-3k+4k^6)})}{k}\left(\frac{\ln(a)}{c\,L\,s\,(\frac{\xi}{2})}\right)^2 \quad (3.17)$$

Where $\zeta = k\,x + \frac{1}{6}\left(9\text{Ln}[a]^4 + 10k^2\text{Ln}[a]^2\sqrt{(-3k+4k^6)\text{Ln}[a]^4} - 20k^5\text{Ln}[a]^4\right)\,t$

4.The description of the homogeneous balance method

suppose we have a nonlinear evolution equation in the form (2.1) Where F is a polynomial in $u(x\,,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. in the following, we give the main steps of this method [8],[11],[13]:

Step .1

Using wave transformation

$$u(x\,,t) = u\,(\zeta)\,, \qquad \zeta = k\,x + \omega\,t \quad (4.1)$$

to reduce equation (2.3) to ODE

Where F is a polynomial in u(ζ) and its total derivatives, while k, ω are constants and $' = \frac{d}{d\zeta}$

Step.2

We suppose that equation (2.3) has the formal solution

$$u(\zeta) = \sum_{n=i}^{N} a_n\, Q(\zeta)^n \quad (4.2)$$

where a_n (n= 0,1,2,3…….,N) are constants to be determined, such that $a_N \neq 0$, and Q(ζ) is the solution of the equation

$$Q'(\zeta) = Q(\zeta)^2 - Q\,(\zeta) \quad (4.3)$$

Equation (4.5) has the solution

$$Q\,(\zeta) = \frac{1}{1 \pm e^{\zeta}} \quad (4.4)$$

Step.3 We determine the positive integer N in equation (4.2) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in equation (2.3).

Step.4 Substitute equation (4.2) into equation (2.3), we calculate all the necessary derivatives u', u'', …… of the function u(ζ). As a result of this substitution, we get polynomial of Q_i , (i = 0,1,2,……..). In this polynomial we gather all terms of same powers and equating them to zero, we obtain a system of algebraic equations which can be solved by the Mathematica or Maple to get the unknown parameters a_n (n=0,1,……., N) , k and ω. Consequently, we obtain the exact solutions of equation (2.1).

5. Application

In this section , we apply the homogeneous balance method to find the exact solutions of the following nonlinear partial differential equations:

Example 1

.the the nonlinear seventh order Sawada-Kotera equation this equation is well known [5], [6],[1], [12],[15]] and has the form

$$u_t + 252u^3u_x + 63(u_x{}^3) + 378uu_xu_{xx} + 126u^2u_{3x} + 63u_{2x}u_{3x} + 42u_xu_{4x} + 21uu_{5x} + u_{7x} = 0 \quad (5.1)$$

let us now solve equation (5.1) by using the homogeneous balance method. to this end, we use the wave transformation (4.2) to reduce equation (5.1) to the following ODE :

$$\omega u' + 252ku^3u' + 63k^3u'^3 + 378k^3uu'u'' + 126k^3u^2u^{(3)} + 63k^5u''u^{(3)} + 42k^5u'u^{(4)} + 21k^5uu^{(5)} + k^7u^{(7)}=0 \quad (5.2)$$

Balancing $u^{(7)}$ with u^3u' yields N= 2. Consequently, equation (5.2) has the formal solution

$$u(\zeta) = a_0 + a_1Q(\zeta) + a_2\,Q(\zeta)^2 \quad (5.3)$$

Where a_0, a_1 *and* a_2 are constants to be determined such that $a_2 \neq 0$. from equation (5.3)

We get

$$u'=(-1 + Q(\zeta))Q(\zeta)(a_1 + 2Q(\zeta)a_2) \quad (5.4)$$

$$u''=(-1 + Q(\zeta))Q(\zeta)[(-1 + 2Q(\zeta))a_1 + 2Q(\zeta)(-2 + 3Q(\zeta))a_2] \quad (5.5)$$

$$u^{(3)} =(-1 + Q(\zeta))Q(\zeta)((1 - 6Q(\zeta) + 6Q(\zeta)^2)a_1 + 2Q(\zeta)(4 - 15Q(\zeta) + 12Q(\zeta)^2)a_2) \quad (5.6)$$

$$u^{(4)}= (-1 + Q(\zeta))Q(\zeta)[(-1 + 14Q(\zeta) - 36Q(\zeta)^2 + 24Q(\zeta)^3)a_1 + 2Q(\zeta)(-8 + 57Q(\zeta) - 108Q(\zeta)^2 + 60Q(\zeta)^3)a_2] \quad (5.7)$$

$$u^{(5)}= (-1 + Q(\zeta))Q(\zeta)[(1 - 30Q(\zeta) + 150Q(\zeta)^2 - 240Q(\zeta)^3 + 120Q(\zeta)^4)a_1 + 2Q(\zeta)(16 - 195Q(\zeta) + 660Q(\zeta)^2 - 840Q(\zeta)^3 + 360Q(\zeta)^4)a_2]$$

(5.8)

$$u^{(6)}=(-1 + Q(\zeta))Q(\zeta)[(-1 + 62Q(\zeta) - 540Q(\zeta)^2 + 1560Q(\zeta)^3 - 1800Q(\zeta)^4 + 720Q(\zeta)^5)a_1 + 2Q(\zeta)(-32 + 633Q(\zeta) - 3420Q(\zeta)^2 + 7500Q(\zeta)^3 - 7200Q(\zeta)^4 + 2520Q(\zeta)^5)a_2]$$

(5.9)

$$u^{(7)}=(-1 + Q(\zeta))Q(\zeta)[(1 - 126Q(\zeta) + 1806Q(\zeta)^2 - 8400Q(\zeta)^3 + 16800Q(\zeta)^4 - 15120Q(\zeta)^5 + 5040Q(\zeta)^6)a_1 + 2Q(\zeta)(64 - 1995Q(\zeta) + 16212Q(\zeta)^2 - 54600Q(\zeta)^3 + 88200Q(\zeta)^4 - 68040Q(\zeta)^5 + 20160Q(\zeta)^6)a_2] \quad (5.11)$$

Substituting (5.3) – (5.11) into (5.2) and equating all the coefficients of powers of $Q(\zeta)$ to zero, we obtain some algebraic equations

Solving the system of the algebraic equations by using the Mathematica or Maple we obtain

Case.1

$$a_1 = 2k^2,\ a_2 = -2k^2,\ \omega = -k^7 - 21k^5a_0 - 126k^3a_0^2 - 252ka_0^3 \quad (5.12)$$

The solution of equation (5.1) is

$$u_1(x,t) = a_0 + \frac{k^2}{2}\mathrm{Sech}^2\left(\frac{k}{2}\,x - \frac{(-k^7-21k^5a_0-126k^3a_0^2-252ka_0^3\,)}{2}\,t\,\right) \quad (5.13)$$

$$u_2(x,t) = a_0 - \frac{k^2}{2}\mathrm{Csch}^2\left(\frac{k}{2}\,x - \frac{(k^7+21k^5a_0+126k^3a_0^2+252ka_0^3)}{2}\,t\,\right) \quad (5.14)$$

Case. 2

$$a_0 = -\frac{k^2}{3},\ a_1 = 4k^2,\ a_2 = -4k^2,\ \omega = \frac{4k^7}{3} \quad (5.15)$$

The solution of equation (5.1) is

$$u_3(x,t) == -\frac{k^2}{3} + k^2\,\mathrm{Sech}^2\left(\frac{k}{2}\,x - \frac{2k^7}{3}\,t\,\right) \quad (5.16)$$

$$u_4(x,t) = -\frac{k^2}{3} - k^2\,\mathrm{Csch}^2\left(\frac{k}{2}\,x - \frac{2k^7}{3}\,t\,\right) \quad (5.17)$$

Example 2.

the nonlinear seventh order Lax equation

This is well known equation [1],[5],[6] and has the form

$$u_t + 140u^3u_x + 70(u_x{}^3) + 280uu_xu_{xx} + 70u^2u_{3x} + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x} = 0 \quad (5.18)$$

let us solve equation(5.18) by using the homogeneous balance method . To this end , me use the wave transformation $\zeta = k\,x + \omega\,t$ to reduce equation (5.18) to the following ODE:

$$\omega u' + 140ku^3u' + 70k^3u'^3 + 280k^3uu'u'' + 70k^3u^2u^{(3)} + 70k^5u''u^{(3)} + 42k^5u'u^{(4)} + 14k^5uu^{(5)} + k^7u^{(7)} = 0 \quad (5.19)$$

Balancing $u^{(7)}$ with u^3u'yields N= 2. Consequently, equation (5.18) has the formal solution (5.3). Substituting (5.3) – (5.8) in to (5.19) and equating all the coefficients of powers of Q(ζ) to zero.

Solving the system of equations ,using the Mathematica or Maple we obtain

Case.1

$$a_1 = 2k^2 \ ,a_2 = -2k^2,\omega = -k^7 - 14k^5a_0 - 70k^3a_0^2 - 140ka_0^3 \quad (5.20)$$

The solution of equation (5.18) is

$$u_1(x,t) = a_0 + \frac{k^2}{2}\text{Sech}^2(\frac{k}{2}\ x - \frac{(k^7+14k^5a_0+70k^3a_0^2+140ka_0^3)}{2}t\,) \quad (5.21)$$

$$u_2(x,t) = a_0 - \frac{k^2}{2}\text{Csch}^2(\frac{k}{2}\ x - \frac{(k^7+14k^5a_0+70k^3a_0^2+140ka_0^3)}{2}t\,) \quad (5.22)$$

Case.2

$$a_0 = \frac{1}{10}i(5ik^2 + \sqrt{5}k^2),a_1 = 6k^2,\ a_2 = -6k^2\ ,\ \omega = -\frac{1}{5}i(-5ik^7 + 21\sqrt{5}k^7) \quad (5.23)$$

The solution of equation (5.18) is

$$u_3(x,t) = \frac{1}{10}i(5ik^2 + \sqrt{5}k^2) + \frac{6}{4}k^2Sech^2(\frac{k}{2}x - \frac{1}{10}i(-5ik^7 + 21\sqrt{5}k^7)t\,) \quad (5.24)$$

$$u_4(x,t) = \frac{1}{10}i(5ik^2 + \sqrt{5}k^2) - \frac{6}{4}k^2Csch^2(\frac{k}{2}x - \frac{1}{10}i(-5ik^7 + 21\sqrt{5}k^7)t\,) \quad (5.25)$$

6.Physical explanations of our obtained solutions

In this section we have presented some graphs of these solutions by taking suitable values of involved unknown parameters to visualize the underlying mechanism of the original equations. Using mathematical software Maple or Mathematica , the plots of some obtained solutions of equations (3.1), (3.10), (5.1) and (5.18) have been shown in Figs. 1-4.

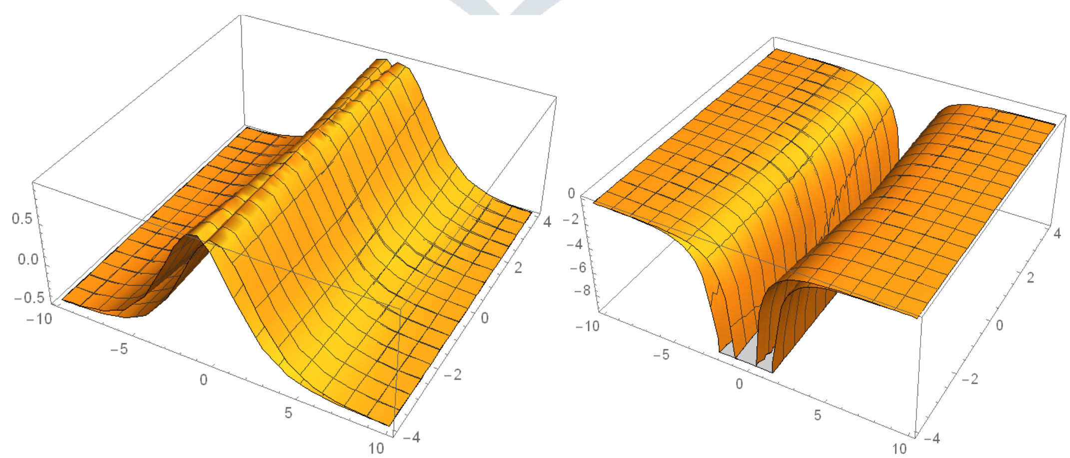

Figure 1: the plot of the solutions (3.5) and (3.6) , when k= 1

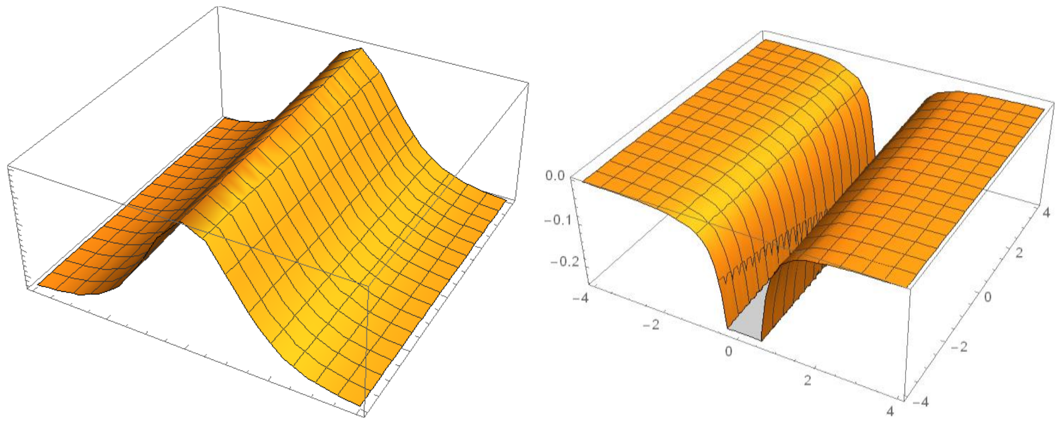

Figure 2: the plot of the solutions (3.13) and (3.14) , when k= 1

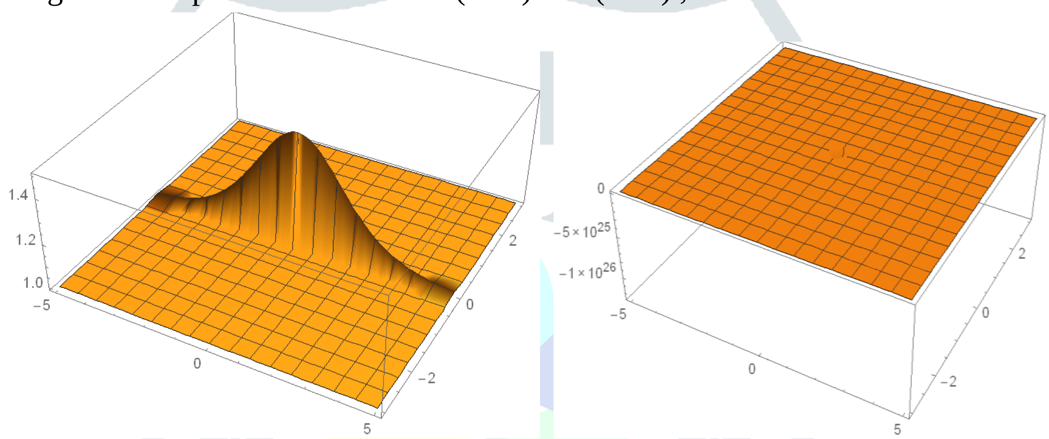

Figure 3: the plot of the solutions (5.21) and (5.22) , when k= 1, $a_0 = 0$

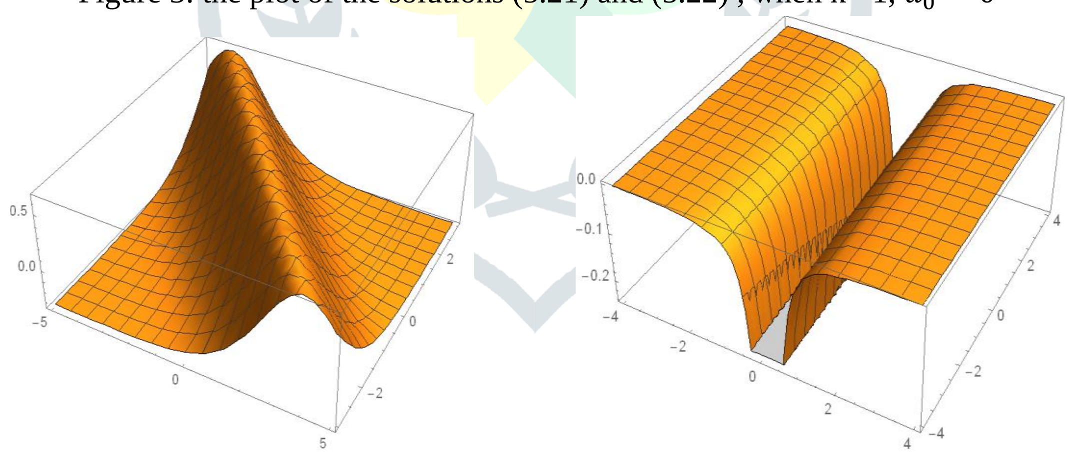

Figure 4: the plot of the solutions (5.16) and (5.17) , when k= 1, $a_0 = 0$

7.Conclusion

We applied two important methods which attract the attention of many authors , the modified Kudryshov method and the homogenous balance method to find new solutions for some higher-order Kdv equations .

These methods were used before but on other equations, as did E. M. Zayed, and K. A. E. Alurrfi in the use of The modified Kudryashov Method in solving the equations of the seventh order and use of the homogenous balance method in solving the equation of fifth order. We have done the opposite in this research.
We used modified Kudryshov method to get exact solution for the nonlinear fifth-order Sawada- Kotera equation and the nonlinear fifth –order lax equation , we find that method is more simpler than other methods and it can be applied to many other nonlinear evolution equation. The homogeneous balance method has been applied to the nonlinear seventh-order Sawada-Kotera equation and the nonlinear seventh-order lax equation . also we deduce that the homogeneous balance method is direct and effective.
And when comparing these solutions with research[1],[8] we have found that the solutions we have obtained are only different versions of the soliton solutions. The similarity of these solutions appears in the diagram.

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