Triple Sequences and Triple Series

[1]Mr. M. Umapathi,[2]Dr.N.Srinivasan,
[1]Research Scholar, Assistant Professor,[2]Professor & Head,
[1]Mathematics Department ,
[1]SPIHER, Avadi, Chennai-54,India

Abstract: In this paper triple sequences and triple series method for finding aCauchy,monotone triple sequences for proof the theorem. This method finds a triple sequence and triple series in the feasible proof. In this method the number of allocation n, m, l is satisfied for all theorems for example double sequence and double series method. This method does not require the theorem. It is easy to understand and this theorem is very efficient for those who are dealing with triple sequence and triple series. It can easily adapt an existing theorem.

***IndexTerms* -** Cauchy Convergence Criterion for triple sequences, Monotone Convergence Theorem.

I.INTRODUCTION

The triple sequences and triple series is the special case of double sequence and double series theorem. It an important role in Cauchy and monotone triple sequences and triple series.

I. The triple sequence *s(n, m, l)* of complex numbers is called Cauchy sequence

II. Monotone triple sequences increasing and decreasing the triple sequence of real numbers of the monotone converges theorem for such sequence that is parallel to their counterparts for single sequences.

1. CAUCHY TRIPLE SEQUENCES

We present in this section the important Cauchy criterion forconvergence of triple sequences.

Definition.

A triple sequence $s(n,m,l)$ of complex numbers is called Cauchy Sequence if and only if for every $\varepsilon > 0$, there exists a natural number $N= N(\varepsilon)$such that

$$|s(p,q,r) - s(n,m,l)| < \varepsilon, \forall\, p \geq n \geq N, q \geq m \geq N, \quad r \geq l \geq N.$$

Theorem : (Cauchy Convergence Criterion for triple sequences)

Statement :

A triple sequence $s(n,m,l)$of complex numbers converges if and only if it is a Cauchy sequence.

Proof :

($\Rightarrow$): Assume that$s(n,m,l) \to a$ as $n,m,l \to \infty$. then given $\varepsilon > 0$,there exists $N \in N$ such that

$$|s(n,m,l) - a| < {}^{\varepsilon}/_{2} \qquad \forall\, n,m,l \geq N.$$

Hence $\forall\, p \geq n \geq N, p \geq m \geq N, r \geq l \geq N$ we have

$$\begin{aligned} |s(p,q,r) - s(n,m,l)| &= |s(p,q,r) - a + a - s(n,m,l)| \\ &\leq |s(p,q,r) - a| + |a - s(n,m,l)| \\ &\leq {}^{\varepsilon}/_{2} + {}^{\varepsilon}/_{2} \\ &< {}^{2\varepsilon}/_{2} \\ &< \varepsilon. \end{aligned}$$

That is$s(n,m,l)$ is Cauchy sequence.

(⇐): Assume that $s(n,m,l)$ is a Cauchy sequence and let $\varepsilon > 0$,be given taking $m = l = n$ and write $s(n,n,n) = b_n$, we see that there exist$k \in N$ such that

$$|b_p - b_n| < \varepsilon \forall \ p \geq n \geq N.$$

Therefore, by Cauchy's criterion for a singlesequence, the sequence (b_n) converges, say to a∈ C.Hence, there exists $N_1 \in N$ such that

$$|b_n - a| < \varepsilon/2, \ \forall \, n \geq N_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since $s(n,m,l)$ is a Cauchy sequence, there exists $N_2 \in N$ such that

$$|s(p,q,r) - b_n| < \varepsilon/2, \ \forall \, p,q,r \geq n \geq N_2 \ldots\ldots\ldots\ldots\ldots (2)$$

Let $N = max\{N_1, N_2\}$ and choose$n \geq N$. Then by (1) and (2) we have

$$\begin{aligned} |s(p,q,r) - a| &\leq |s(p,q,r) - b_n| + |b_n - a| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\leq 2\varepsilon/2 \\ &< \varepsilon . \qquad \forall \, p,q,r \geq N \end{aligned}$$

Hence $s(n,m,l)$ converges to 'a'.

2 MONOTONE TRIPLE SEQUENCES

In this section, we define increasing and decreasing triple sequences of a real number and weprove amonotone convergence theorem for such sequencesthat are parallel to their counterparts for single sequences.

2.1Definition:

Let $s(n,m,l)$ be a triple sequence of real numbers

I. If $s(n,m,l) \leq s(i,j,k) \forall \, (n,m,l) \leq (i,j,k)$ in $N \times N \times N$, we say the sequence is increasing.

II. If $s(n,m,l) \geq s(i,j,k) \forall \, (n,m,l) \leq (i,j,k)$ in $N \times N \times N$, we say the sequence is decreasing.

III. If $s(n,m,l)$is either increasing (or) decreasing, then we sayit is monotone.

The sequence 1, $1\frac{1}{2}$, $1\frac{63}{64}$, $1\frac{511}{512}$,……………….. (that is $\{2 - \frac{1}{2^{n-1} m^{-1} l^{-1}}\}_{n,m,l=1}^{\infty}$) is non decreasing (and bounded). The sequence $\{nml\}_{n,m,l=1}^{\infty}$ is non decreasing (and not bounded).

2.2 Theorem: (Monotone Convergence Theorem)

Statement

A monotone triple sequence of real numbers isconvergent if and only if it is bounded. Further:

a. If $s(n,m,l)$ is increasing and bounded above then

$$\begin{aligned} \lim_{n \to \infty} \left\{ \lim_{m,l \to \infty} s(n,m,l) \right\} &= \lim_{m \to \infty} \left\{ \lim_{l,n \to \infty} s(n,m,l) \right\} \\ &= \lim_{l \to \infty} \left\{ \lim_{m,n \to \infty} s(n,m,l) \right\} \\ &= \lim_{n,m,l \to \infty} s(n,m,l) \end{aligned}$$

$$= sup\{\ s(n,m,l): n,m,l \in N\ \}$$

b. If $s(n,m,l)$ is decreasing and bounded below, then

$$\lim_{n\to\infty}\left\{\lim_{m,l\to\infty} s(n,m,l)\right\} = \lim_{m\to\infty}\left\{\lim_{l,n\to\infty} s(n,m,l)\right\}$$

$$= \lim_{l\to\infty}\left\{\lim_{m,n\to\infty} s(n,m,l)\right\}$$

$$= \lim_{n,m,l\to\infty} s(n,m,l)$$

$$= inf\{s(n,m,l): ,m,l \in N\ \}$$

Proof:

Assume that a monotone triple sequence of real numbers is convergent.

$\xrightarrow{\text{TPT}}$ The convergent sequence must be bounded

Suppose $s(n,m,l) \to a$ and let $\varepsilon = 1$.then there exists $N\in N$ such that

$$n,m,l \geq N \quad \Rightarrow |s(n,m,l) - a| < 1$$

This and the triangle inequality yield that $|s(n,m,l)| < 1 + |a| \quad \forall\, n,m,l \geq N.$

Let $M = max\ \{\,|S(1,1,1)|\,, |S(1,1,2)|\,, |S(1,2,1)|\,, |S(2,1,1\,)|\ldots\ldots\ldots\ldots |S(N-1,N-1,N-1\,)|, |a| + 1\,\}$

Clearly $|s\,(\,n,m,l\,)| \leq M\ \forall\, n,m,l\ \geq N$

Conversely:

Let $s(\,n,m,l\,)$ the bounded monotone sequence then $s(\,n,m,l\,)$ is increasing or decreasing.

(a) We first treat the case that $s(\,n,m,l\,)$ is increasing and bounded above.

By the supremum principle of real numbers, the supremum $a^* = sup\{\ s(n,m,l) :\ n,m,l \in N\,\}$ exists.

We shall show that the triple and iterated limits of $s(\,n,m,l\,)$ exists and are equal to a^*

It $\varepsilon > 0$ is given, then $a^* - \varepsilon$ is not an upper bound for the set

$$\{s(n,m,l): n,m,l \in N\,\};$$

hence there exists natural numbers $k(\varepsilon), j(\varepsilon)\ and\ i(\varepsilon)$ such that

$$a^* - \varepsilon < s(i,j,k) \leq\ s(n,m,l) \leq a^* < a^* + \varepsilon. \forall\, n,m,l\ \geq i,j,k.$$

And hence $\quad |\,s\,(\,n,m,l\,) - a^*| < \varepsilon\forall\, n,m,l\ \geq i,j,k.$

Since $\varepsilon > 0$ was arbitrary, itfollows that $s(n,m,l)$ converges to a^*

Next to show that $\lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty} s(n,ml)\right]\right\} = \lim_{n,m,l\to\infty} s(n,m,l) = a^*$

Note that Since $s(\,n,m,l\,)$ is bounded above,

Then, for each fixed $m\in N$, the single sequence $\{s(n,m,l) : l \in N\,\}$ is bounded above,and increasing. So by monotone convergence for single sequences, we have

$$\lim_{l\to\infty} s(n,m,l) = sup\{s(n,m,l): l \in N\} = l_{nm} \qquad \forall\, n,m\ \in N$$

Hence by theorem *{2.11,2.12,2.13}*

The iterated limits exist and

$$\lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty} s(n,ml)\right]\right\} = \lim_{n,m,l\to\infty} s(n,m,l) = a^*$$

$$\lim_{m\to\infty}\left\{\lim_{l\to\infty}\left[\lim_{n\to\infty} s(n,ml)\right]\right\} = \lim_{n,m,l\to\infty} s(n,m,l) = a^*$$

Similarly, it can be shown that

$$\lim_{l\to\infty}\left\{\lim_{n\to\infty}\left[\lim_{m\to\infty} s(n,ml)\right]\right\} = \lim_{n,m,l\to\infty} s(n,m,l) = a^*$$

(b) If $s(n,m,l)$ is decreasing and bounded below, then the sequence $[-s(n,m,l)]$ is Increasing and bounded above. Hence by the part (a), we obtain

$$\lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty} - s(n,m,l)\right]\right\} = \lim_{m\to\infty}\left\{\lim_{l\to\infty}\left[\lim_{n\to\infty} - s(n,m,l)\right]\right\}$$

$$= \lim_{l\to\infty}\left\{\lim_{n\to\infty}\left[\lim_{m\to\infty} - s(n,m,l)\right]\right\}$$

$$= \lim_{n,m,l\to\infty} - s(n,m,l)$$

$$= sup\{-s(n,m,l): n,m,l \in N\}$$

$$= -inf\{s(n,m,l): n,m,l \in N\}$$

It follows that

$$\lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\lim_{l\to\infty} s(n,m,l)\right]\right\} = \lim_{m\to\infty}\left\{\lim_{l\to\infty}\left[\lim_{n\to\infty} s(n,m,l)\right]\right\}$$

$$= \lim_{l\to\infty}\left\{\lim_{n\to\infty}\left[\lim_{m\to\infty} s(n,m,l)\right]\right\}$$

$$= \lim_{n,m,l\to\infty} s(n,m,l)$$

$$= inf\{s(n,m,l): n,m,l \in N\}.$$

3 Triple series

In this section, we introduce triple series and we shall give the definition of their convergence and divergence. Then we study the relationship between triple and iterated series, and we give a sufficient condition for equality of iterated series.

Definition:

Let $z: N \times N \times N \to C$ be a triple sequence of complex numbers and let $s(n,m,l)$ be the triple sequence defined by the equation $s(n,m,l) = \sum_{i=1}^{n}\{\sum_{j=1}^{m}[\sum_{k=1}^{l} z(i,j,k)]\}$ the pair (z,s) is called a triple series. And is denoted by the symbol $\sum_{n,m,l=1}^{\infty} z(n,m,l)$ or $\sum z(n,m,l)$. Each number $z(n,m,l)$ is called a term of the triple series and each $s(n,m,l)$ is called a partial sum.

Theorem

If the triple series $\sum_{n,m,l=1}^{\infty} z(n,m,l)$ is convergent then $\lim_{n,m,l\to\infty} z(n,m,l) = 0$

Proof

Since the triple series $\sum_{n,m,l=1}^{\infty} z(n,m,l)$ is convergent, say to S, Then its sequence of partial sums $s(n,m,l)$ converges to S. So given $\varepsilon > 0$, there exists $N \in N$ such that $|s(n,m,l) - S| < \varepsilon/8 \ \forall\, n,m,l \geq N$ converges to S.

It follows that for all $\forall\, n,m,l \geq N$, we have

$$|z(n,m,l)| \leq |s(n,m,l) - S| + |s(n-1,m-1,l-1) - S| + |s(n,m,l-1) - S| + |s(n,m-1,l) - S|$$
$$+|s(n-1,m,l) - S| + |s(n,m-1,l-1) - S| + |s(n-1,m,l-1) - S| + |s(n-1,m-1,l) - S|$$
$$< \varepsilon/8 + \varepsilon/8 + \varepsilon/8 + \varepsilon/8 + \varepsilon/8 + \varepsilon/8 + \varepsilon/8 + \varepsilon/8$$
$$< 8\varepsilon/8 < \varepsilon.$$

CONCLUSION:

In this paper, a new triple sequence and triple series were introduced. Cauchy sequence and monotone sequence proof the theorems. The double sequences and double series while comparing to triple sequences and triple series. It is easy to proof and we get the required proof of theorems.

REFERENCES

[1] T. M. Apostol, Mathematical Analysis, Second edition, Addison Wesley,1974.

[2] R. G. Bartle and D. R. Sherbert, Introduction to Real Analysis, Second edition, Wiley and Sons, 1992.

[3] W. W. L. Chen, Fundamentals of Analysis, Published by W. W. L. Chen viaInternet, 2003.

[4] A. Garcia-Martinez, Webbed spaces, double sequences, and the Mackey convergence condition, Internat. J. Math. and Math. Sci., Vol 22, No. 3(1999),521-524.

[5] S. L. Gupta and N. Rani, Principles of Real Analysis, Vikas Publishing House, New Delhi, 1998.

[6] W. Rudin, Principles of Mathematical Analysis, Third Edition, McGraw-Hill,1976.

[7] K. R. Stromberg, An Introduction to Classical Real Analysis, Wadsworth,1981.