LOCAL EXISTENCE OF NONLINEAR FUZZY FUNCTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSE EFFECT

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Abstract : Fuzzy functional differential equations play important role in capturing the uncertainty of various dynamical systems. On the other hand, impulsive differential equations help in modeling the instantaneous disturbances. Here in this paper, we are considering a fuzzy functional differential equation with impulse effect. The aim is to establish the local existence result for a nonlinear fuzzy functional differential equation with impulses. The existence result is proved to make use of the method of successive approximations.

***Index Terms* - Fuzzy metric spaces・Fuzzy differential equations・Impulsive differential equations・Functional differential equations・Local existence.**

I. INTRODUCTION

Functional differential equations with or without impulses plays an important role in modeling the dynamics of a number of natural and artificial processes in diverse field of application. But, when the deterministic ordinary differential equations are used to model a dynamical system, we donot have surety that the model is perfect because, in general, the dynamical systems are uncertain and so the knowledge about dynamical system, in general, is incomplete and vague. So there is a need to inculcate vagueness in the model and that could be done by making use of fuzzy set theory. Even after inserting vagueness into the model, it may not be perfect.

For example, consider the following Fuzzy Time-Delay Malthusian Model developed by Lupulescu, V., (2009):

$$N'(t) = rN(t), \quad t \geq 0,$$

$$N(t) = N_0, \quad -\sigma \leq t \leq 0,$$

where $[N_0]^\alpha = (1-\alpha)[-1,1]$, $\alpha \epsilon [0,1]$ and r denotes the rate at which the population grows at time t and also it depends on population at time $t-\sigma$, σ is a positive number.

Now, this model may or may not be perfect because in real situations, the population dynamics can be influenced instantly by many factors like earthquake, war, pestilence, tsunami etc, which may decrease the population too large extent within a small interval of time and will affect the next population growth. So there is a need to include such instantaneous changes in the form of impulses in the population growth model. Therefore, fuzzy delay Malthusian model with impulse effect would be a more realistic approach to describe the population growth.

Recently an adequate amount of work is going on for the fuzzy functional differential equations with impulse effect. Guo et al. (2003) studied fuzzy functional impulsive differential equations. They established some existence results using Hullermeier approach and these results were then used to study fuzzy population models. Chalishajar et al. (2017) established some existence results for the solutions of fuzzy impulsive neutral functional differential equations. The results are proved by making use of Banach fixed point theorem.

II. Basic Concept

The concept of fuzzy set is based on the idea that each element x in the base set X is assigned a membership grade $a(x)$ taking values in [0,1]. According to Zadeh, a fuzzy subset of X is a nonempty subset $\{(x, a(x)): x \in X\}$ of $X \times [0,1]$ for some function $a: X \to [0,1]$. In particular, a fuzzy subset of $\mathfrak{R}^n$ is defined in terms of a membership function which assigns to each point $x \in \mathfrak{R}^n$ a grade of membership in fuzzy set. Such a membership function $a: \mathfrak{R}^n \to [0,1]$ is used to denote the corresponding fuzzy set.
For each $\alpha \in (0,1]$, the α-level set $[a]^\alpha$ of fuzzy set a is subset of those $x \in \mathfrak{R}^n$ whose membership grade $a(x)$ is atleast α, that is

$$[a]^\alpha = \{x \in \mathfrak{R}^n: a(x) \geq \alpha\}.$$

Denote

$E^n = \{a: \mathfrak{R}^n \to [0,1]$ / a satisfy the following four conditions}

(i) a is normal i.e. there exist an $x_0 \in \mathfrak{R}^n$ such that $a(x_0) = 1$;
(ii) $[a]^0 = x \in \mathfrak{R}^n: a(x) > 0$ is compact;
(iii) a is upper semi-continuous;
(iv) a is fuzzy convex, i.e. $a(\lambda x + (1-\lambda)y) \geq min\{a(x), a(y)\}, 0 \leq \lambda \leq 1$.

From (i) to (iv), it is clear that $[a]^\alpha \in P_K(\mathfrak{R}^n)$, where $P_K(\mathfrak{R}^n)$ denotes the collection of all nonempty, convex compact subsets of $\mathfrak{R}^n$.
We define $\hat{0} \in E^n$ as $\hat{0}(x) = 1$ if $x = 0$ and $\hat{0}(x) = 0$ if $x \neq 0$.
It is well known that (E^n, D) is a complete metric space with the metric D defined by

$$D[a,b] = \sup_{0 \leq \alpha \leq 1} d_H([a]^\alpha, [b]^\alpha), \forall a, b \in E^n,$$

where d_H is a Hausdroff metric distance between the level sets of fuzzy sets.
If there exist $c \in E^n$ such that $a = b + c$, then c is called the H-difference of a and b and is denoted by $a - b$.
Let I be an interval of $\mathfrak{R}$. A mapping $F: I \to E^n$ is differentiable at $t_0 \in I$ if for small $h > 0$, there exist H −differences $F(t_0 + h) - F(t_0)$, $F(t_0) - F(t_0 - h)$ and there exists $F'(t_0) \in E^n$ such that the limits $\lim_{h \to 0^+} \frac{F(t_0+h)-F(t_0)}{h}$ and $\lim_{h \to 0^+} \frac{F(t_0)-F(t_0-h)}{h}$ exist and equal to $F'(t_0)$.
The fuzzy-valued function $F: I \to E^n$ is called strongly measurable, if for each $\alpha \in [0,1]$, the set valued function $F_\alpha: I \to P_K(\mathfrak{R}^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable. The integral of F over I, denoted by $\int_I F_\alpha(t)dt$ is defined level-wise by the equation
$[\int_I F(t)dt]^\alpha = \int_I F_\alpha(t)dt$, for all $0 \leq \alpha \leq 1$.
A strongly measurable and integrally bounded function $F: I \to E^n$ is said to be integrable over I if $\int_I F_\alpha(t)dt \in E^n$.
For a positive number σ, let $C_\sigma = C([-\sigma, 0], E^n)$ denotes the space of fuzzy continuous functions from $[-\sigma, 0]$, to E^n. Also, we denote by

$$D_\sigma = \sup_{t \in [-\sigma,0]} D[u(t), v(t)],$$

the metric on the space C_σ.

III. Local Existence Result

For a positive number σ, let $PC_\sigma = PC([-\sigma, 0], E^n)$ denotes the space of piecewise continuous fuzzy functions from $[-\sigma, 0]$ to E^n. Also, we denote by D_σ, the metric on PC_σ defined as

$$D_\sigma = \sup_{t \in [-\sigma,0]} D[u(t), v(t)].$$

For a given constant $\rho > 0$, let $B_\rho = \{\phi \in PC_\sigma : D_\sigma(\phi, \hat{0}) \leq \rho\}$.

Consider nonlinear fuzzy functional differential equation with impulse effect

$$u'(t) = F(t, u_t), \ t \neq t_k,$$

$$u(t_k^+) = u(t_k) + I_k(u(t_k)), k = 1,2, \ldots, \qquad (1)$$

$$u(t) = \varphi(t - t_0), \ t_0 - \sigma \leq t \leq t_0,$$

where $F: J \times PC_\sigma \to E^n$, $J = [0, \infty)$, $I_k: E^n \to E^n$, $t_k, k = 1,2, \ldots$ are the points of impulses such that $t_k < t_{k+1}$ and $t_k \to \infty$ as $k \to \infty$. Let $J^* = [t_0 - \sigma, \infty)$ and

$$PC(J^*, E^n) = \begin{cases} u: u \text{ is a piecewise continuous map from } [t_0 - \sigma, \infty) \text{ into } E^n \\ \text{such that } u(t) \text{ is left continuous at } t = t_k \text{ and } u(t_k^+) \text{ exists,} \\ k = 1, 2, \ldots, u(t) = \varphi(t - t_0), t_0 - \sigma \le t \le t_0 \text{with } u(t_0) = \varphi(0). \end{cases}$$

Then $PC(J^*, E^n)$ is a metric space and a map $u \in PC(J^*, E^n)$ called a solution of (1), if it satisfies (1).

Now we proceed to prove local existence theorem for fuzzy functional differential equation with impulse effect using the method of successive approximation.

Definition 3.1 A function $F: J \times PC_\sigma \to E^n$ is said to be locally Lipschitz if for all $a, b \in J$ and $\rho > 0$, there exists $L > 0$ such that

$$D[F(t,\varphi), F(t,\psi)] \le LD_\sigma(\varphi,\psi), \quad \varphi, \psi \in B_\rho, a \le t \le b,$$

where B_ρ denotes a ball with radius ρ.

Lemma 3.2 Assume that $F: J \times PC_\sigma \to E^n$ is locally Lipschitz and jointly continuous function. Let $I \subset J$ be any compact interval and $\rho > 0$, then there exists $K > 0$ such that

$$D\big[F(t,\varphi), \hat{0}\big] \le K, \quad t \in I, \varphi \in B_\rho.$$

Proof: For $t \in I$, we have

$$D\big[F(t,\varphi), \hat{0}\big] \le D\big[F(t,\varphi), F\big(t,\hat{0}\big)\big] + D\big[F\big(t,\hat{0}\big), \hat{0}\big]$$

$$\le LD_\sigma\big(\varphi, \hat{0}\big) + D\big[F\big(t,\hat{0}\big), \hat{0}\big] \le \rho L + \eta = K,$$

where $\eta = sup_{t\in I} D\big[F\big(t,\hat{0}\big), \hat{0}\big]$.

Theorem 3.3 Assume that $F: J \times PC_\sigma \to E^n$ is jointly continuous and locally Lipschitz. Further suppose that

(i) for every $k = 1, 2, \ldots, r$, $\exists$ nonnegative number b_k such that for any $u, v \in E^n$,

$$D(I_k(u), I_k(v)) \le b_k D(u,v);$$

(ii) $\alpha = max\{b_k : 1 \le k \le r\} < 1$;

(iii) $\exists N_k > 0$ such that $D(I_k, \hat{0}) \le N_k, k = 1, 2, \ldots, r$ and $\Sigma N_k = N < \infty$.

Then the following fuzzy functional differential equation with impulse effect

$$\begin{aligned} u'(t) &= F(t, u_t), \quad t \ne t_k, t \in [t_0, T], \\ u(t_k^+) &= u(t_k) + I_k\big(u(t_k)\big), k = 1, 2, \ldots, r, \\ u(t) &= \varphi(t - t_0), \quad t_0 - \sigma \le t \le t_0, \end{aligned} \tag{2}$$

has a solution.

Proof: Let $\rho > o$ be positive. Since F is locally Lipchitz, there exists $L > 0$ such that

$$D[F(t,\varphi), F(t,\psi] \le LD_\sigma(\varphi,\psi), \quad t_0 \le t \le h, \ \varphi, \psi \in B_{2_p}, \tag{3}$$

for some $h > t_0$. By Lemma 3.2, there exists $K > 0$ such that $D\big[F(t,\varphi), \hat{0}\big] \le K$ for $(t,\varphi) \in [t_0, h] \times B_{2\rho}$.

Let $T = min\left\{h, \frac{\rho-N}{K}\right\}$.

We see that if $u \in E^n$, then we can define a piecewise continuous function $w: [t_0-\sigma, T] \to E^n$ by

$$w(t) = \begin{cases} \varphi(t-t_0), & t_0-\sigma \le t \le t_0, \\ \varphi(0) + \int_{t_0}^{t} F(s,u_s)\,ds + \sum_{t_0 \le t_k \le T} I_k(u(t_k)), & t_0 \le t \le T. \end{cases}$$

Then for $t \in [t_0, T]$, we have

$$D[w(t),\hat{0}] \le D[\varphi(0),\hat{0}] + D[\int_{t_0}^{t} F(s,u_s)\,ds, \hat{0}] + D[\sum_{t_0 \le t_k \le t} I_k(u(t_k)), \hat{0}]$$

$$\le \rho + \int_{t_0}^{t} D[F(s,u_s),\hat{0}]ds + \sum_{t_0 \le t_k \le t} D[I_k(u(t_k)),\ \hat{0}]$$

$$\le \rho + KT + N \ \le\ 2\rho,$$

and so $w \epsilon E^n$. To solve (2) we shall apply the method of successive approximations, constructing a sequence of piecewise continuous functions $u^m: [t_0-\sigma, T] \to E^n$ starting with the initial function

$$u^0(t) = \begin{cases} \varphi(t-t_0), & for\ t_0-\sigma \le t \le t_0, \\ \varphi(0), & for\ t_0 \le t \le T. \end{cases}$$

Clearly, $D[u^0(t),\hat{0}] \le \rho\ on\ [t_0, T]$. Further, for $m = 0, 1, 2, \ldots,$ we define

$$u^{m+1}(t) = \begin{cases} \varphi(t-t_0), & for\ t_0-\sigma \le t \le t_0, \\ \varphi(0) + \int_{t_0}^{t} F(s,u_s^m)ds + \sum_{t_0 \le t_k \le T} I_k(u^m(t_k)), & for\ t_0 \le t \le T. \end{cases} \quad (4)$$

Then for $t \in [t_0, T]$,

$$D[u^1(t), u^0(t)] \le D\left[\int_{t_0}^{t} F(s,u_s^0)ds, \hat{0}\right] + D[\sum_{t_0 \le t_k \le T} I_k(u^0(t_k)), \hat{0}]$$

$$\le \int_{t_0}^{t} D[F(s,u_s^0),\hat{0}]\,ds + \sum_{t_0 \le t_k \le T} D(I_k(u^0(t_k)),\ \hat{0}) \le K(t-t_0). \quad (5)$$

Using (3), (4) and condition (i), (ii), we get that

$$D[u^{m+1}(t), u^m(t)] \le D\,[\int_{t_0}^{t} F(s,u_s^m)ds, \int_{t_0}^{t} F(s,u_s^{m-1})]ds$$

$$+D[\sum_{t_0 \le t_k \le T} I_k(u^m(t_k)), \sum_{t_0 \le t_k \le T} I_k(u^{m-1}(t_k))]$$

$$\le \int_{t_0}^{t} D[F(s,u_s^m), F(s,u_s^{m-1})]ds + \sum_{t_0 \le t_k \le T} D[I_k(u^m), I_k(u^{m-1})]$$

$$\le \int_{t_0}^{t} LD_\sigma(u_s^m, u_s^{m-1})ds + \sum_{t_0 \le t_k \le T} b_k D(u^m, u^{m-1})$$

$$= \int_{t_0}^{t} L \sup_{r \in [-\sigma,0]} D[u_s^m(r), u_s^{m-1}(r)]\ ds + r\alpha D(u^m, u^{m-1})$$

$$= \int_{t_0}^{t} L \sup_{r \in [-\sigma,0]} D[u^m(s+r), u^{m-1}(s+r)]ds$$

$$+ r\alpha D(u^m, u^{m-1})$$

$$= \int_{t_0}^{t} L \sup_{\theta \in [s-t_0, s]} D[u^m(\theta), u^{m-1}(\theta)]\,ds$$

$$+\ r\alpha D(u^m, u^{m-1}), \quad t \in [t_0, T].$$

In particular, making use of (5), we have

$$D[u^2(t), u^1(t)] \le L \int_{t_0}^{t} K(s - t_0)ds + r\alpha K(t - t_0) = \frac{K}{L}\frac{[L(t-t_0)]^2}{2!} + r\alpha K(t - t_0),$$

$$t \in [t_0, T].$$

Also

$$D[u^3(t), u^2(t)] \le L \int_{t_0}^{t} \{\frac{K}{L}\frac{[L(s - t_0)]^2}{2!} + r\alpha K(s - t_0)\}ds + r\alpha[\frac{K}{L}\frac{[L(t - t_0)]^2}{2!}$$

$$+ r\alpha K(t - t_0)]$$

$$= \frac{K}{L}\frac{[L(t-t_0)]^3}{3!} + 2r\alpha\frac{K}{L}\frac{[L(t-t_0)]^2}{2!} + r^2\alpha^2 K(t - t_0).$$

Continuing, we get

$$D[u^m(t), u^{m-1}(t)] \le \frac{K}{L}\frac{[L(t - t_0)]^m}{m!} + (m - 1)r\alpha\frac{K}{L}\frac{[L(t - t_0)]^{m-1}}{(m - 1)!} + \cdots$$

$$+ r^{m-1}\alpha^{m-1}K(t - t_0), \quad t \in [t_0, T]. \quad (5)$$

Thus ultimately, the series $\sum_{m=1}^{\infty} D[u^m(t), u^{m-1}(t)]$ is uniformly convergent on $[t_0, T]$, and so the sequence $\{u^m\}_{m\ge0}$ is also uniformly convergent. Therefore there exists a piecewise continuous function $u: [t_0, T] \to E^n$ such that $sup_{t_0\le t\le T}\ D[u^m(t), u(t)] \to 0\ as\ m \to \infty$.

Since $D[F(s, u_s^m), F(s, u_s)] \le LD_\sigma(u_s^m, u_s) \le L\, sup_{t_0\le t\le T}\ D[u^m(t), u(t)]$,

And $D(I_k(u_s^m), I_k(u)) \le b_k D(u^m, u)$,

we get that $D[F(s, u_s^m), F(s, u_s)] \to 0$ uniformly on $[t_0, T]$ as $m \to \infty$ and similarly $D(I_k(u_s^m), I_k(u)) \to 0$ uniformly on $[t_0, T]$.

Therefore, since $D\left[\int_{t_0}^{t} F(s, u_s^m)\, ds + \sum_{t_0\le t_k\le T} I_k(u^m),\ \int_{t_0}^{t} F(s, u_s)\, ds + \sum_{t_0\le t_k\le T} I_k(u)\right]$

$$\le \int_{t_0}^{t} D[F(s, u_s^m), F(s, u_s)]ds + \sum_{t_0\le t_k\le T} D(I_k(u_s^m), I_k(u)),$$

it follows that

$$lim_{m\to\infty} \int_{t_0}^{t} F(s, u_s^m)ds + \sum_{t_0\le t_k\le T} I_k(u^m) = \int_{t_0}^{t} F(s, u_s)\ ds + \sum_{t_0\le t_k\le T} I_k(u),, t \in [t_0, T].$$

Extending u to $[t_0 - \sigma, t_0]$ in the usual way by $u(t) = \varphi(t - t_0)$ for $t \in [t_0 - \sigma, t_0]$, then by (4) we obtain that

$$u(t) = \begin{cases} \varphi(t - t_0), & if\ t \in [t_0 - \sigma, t_0], \\ \varphi(0) + \int_{t_0}^{t} F(s, u_s)\, ds + \sum_{t_0\le t_k\le T} I_k(u(t_k)), & if\ t \in [t_0, T], \end{cases}$$

and so u is a solution for (2). This completes the theorem.

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