Posets and Forbidden induced subgraph of the Comparability Graph

[1]Baiju Sukumaran, [2]Antony Mathews, [3]Sibi C Babu,[4]Athul T B
[1]Associate Professor, [2] Associate Professor,[3]Assistant Professor,[4]M.Phil Scholar
[1]Depatment of Mathematics,
[1] Sree Narayana College, Punalur, India

***Abstract: The** cover-incomparability graph of a poset P is the edge-union of the covering and the incomparability graph of P. As a continuation of the study of 3-colored diagrams we characterize some forbidden ⊲ - preserving subposets of the posets whose cover-incomparability graph contains one of the forbidden induced subgraph of the comparability graph.*

***Index Terms**:-Cover-incomparability graph, comparability graph, Poset*

INTRODUCTION

Cover-incomparability graphs of posets, or shortly C-I graphs, were introduced in [2] as the underlying graphs of the standard interval function or transit function on posets (for more on transit functions in discrete structures [3, 4, 5, 6, 11]). On the other hand, C-I graphs can be defined as the edge-union of the covering and incomparability graph of a poset; in fact, they present the only non-trivial way to obtain an associated graph as unions and/or intersections of the edge sets of the three standard associated graphs (i.e. covering, comparability and incomparability graph). In the paper that followed [9], it was shown that the complexity of recognizing whether a given graph is the C-I graph of some poset is in general NP-complete. In [1] the problem was investigated for the classes of split graphs and block graphs and the C-I graph within these two classes of graphs were characterized. This resulted in linear-time recognition algorithms for C-I block and C-I split graphs. It was also shown in [1] that whenever a C-I graph is a chordal graph, it is necessarily an interval graph, however a structural characterization of C-I interval graphs (and thus C-I chordal graphs) is still open. C-I distance-hereditary graphs have been characterized and shown to be efficiently recognizable [10]. Let P = (V ; ≤) be a poset. If u ≤ v but u ≠ v, then we write u < v. For u, v ϵ V we say that v *covers* u in P if u < v and there is no w in V with u < w < v. If u ≤ v we will sometimes say that u is *below* v, and that v is *above* u. Also, we will write u ⊲ v if v covers u; and u ⊲⊲ v if u is below v but not covered by v. By u ∥ v we denote that u and v are incomparable. Let V' be a nonempty subset of V. Then there is a natural poset Q = (V'; ≤ ′), where u ≤ ′ v if and only if u ≤ v for any u, v ϵV'. The poset Q is called a *subposet* of P and its notation is simplified to Q = (V';≤). If, in addition, together with any two comparable elements u and v of Q, a chain of shortest length between u and v of P is also in Q, we say that Q is an isometric subposet of P. Recall that a poset P is *dual* to a poset Q if for any x, y ϵ P the following holds: x ≤ y in P if and only if y ≤ x in Q. Given a poset P, its cover-incomparability graph G_P has V as its vertex set, and uv is an edge of G_P if u ⊲ v, v ⊲ u, or u and v are incomparable. A graph that is a cover-incomparability graph of some poset P will be called a C-I graph.

Lemma 1 [2] Let P be a poset and G_P its C-I graph. Then
(i) G_P is connected;
(ii) vertices in an independent set of G_P lie on a common chain of P;
(iii) an antichain of P corresponds to a complete subgraph in G_P ;
(iv) G_P contains no induced cycles of length greater than 4.

II.3-colored diagrams

A 3-coloured diagram Q in [13] is explained as follows. Let G be a C-I graph and H be an induced subgraph of G. We note that there can be different ⊲- preserving subposets Q_i of some posets with G_{Q_i} isomorphic to the subgraph H. Let u,v,w be an induced path in the direction from u to v in H. There are four possibilities in which u, v and w can be related in the ⊲ - preserving subposets. It is possible to have u ⊲ v, u ∥ v, v ⊲ w and v ∥ w. Each case will appear as a ⊲ - preserving subposet of four different posets. If u ⊲ v and v ⊲ w in a subposet, then u ⊲ v ⊲ w is a chain in the subposet and u,v,w is an induced path in H. If there is either u ∥ v or v ∥ w in a subposet Q, then there should be another chain from u to w in Q in order to have u, v,w an induced path in H. We try to capture this situation using the idea of 3-colored diagram. Suppose in ⊲ - preserving subposet Q of a poset P, there exists two elements u,v which is always connected by some chain of length three in Q. Let w be an element in Q such that either both uw and vw are red edges or any one of them is a red edge. Then in order to have a chain between u and v, there must exist an element x in Q so that u, x, v form a chain in Q. When both edges are normal, then we have the chain u, w, v in Q and hence the chain u, x, v is not required in this case. We denote the chain u, x, v by dashed lines between ux and xv in order to specify that it is possible to have the presence or absence of the chain u, x, v in Q. The presence of the chain u, x, v implies that either both of the edges uw and wv are red edges or one of them is a red edge. The absence of the chain implies that both uw and vw are normal edges in Q. We call posets having the above mentioned diagrams as 3-colored diagrams.

Theorem 2: (Theorem 1,[8]): Let G be a class of graphs with a forbidden induced subgraphs characterization. Let 𝒫= {P | P is a poset with G_{TP} ϵ G }. Then 𝒫 has a characterization by forbidden ⊲ - preserving subposets.

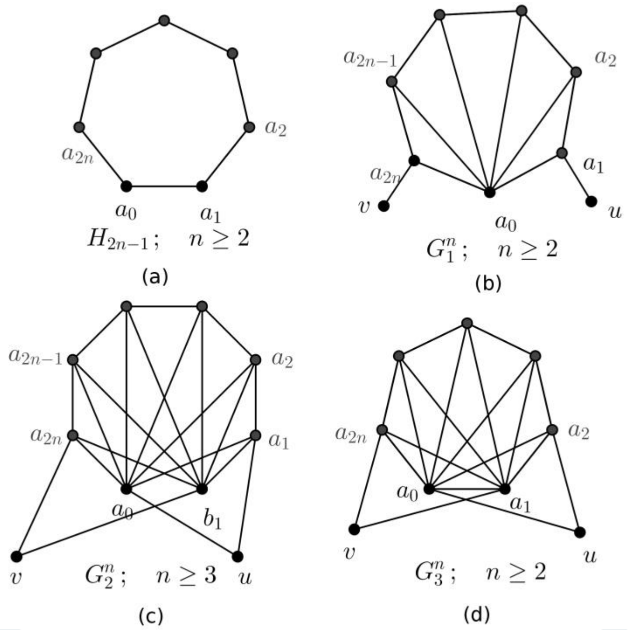

Figure 1: Families of forbidden induced subgraphs for comparability graph[14]

Let $\mathcal{F}$*(G)*be a collection of forbidden induced subgraphs**.** We consider the posets whose cover-incomparability graph **G**$_P$ contains the graphs in $\mathcal{F}$***(G).***

III. ⊲ - preserving subposets of posets whose C-I graphs contains the graphs in $\mathcal{F}$*(G)*

We have the following theorem regarding the graph family $\mathcal{F}$***(G)***

.

Theorem 3: If P is a poset, then G_P contains $\mathcal{F}$**(**G_1^n**),** $n \geq 2$if and only if P contains the 3-colored diagrams Q_1 from Figure 2 and their duals.

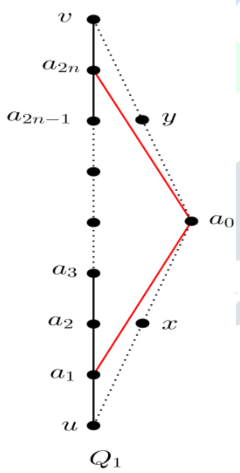

Figure 2:Forbidden 3-colored diagrams for posets whose C-I graphs contains G_1^n, depicted in Figure 1 (b).

Proof.Suppose P contains3-colored diagram Q_1. Then since $a_2, a_3, \ldots, a_{2n-1}$ and a_0 are incomparable in P, the set of vertices { u, a_0, $a_1, a_2, a_3, \ldots, a_{2n-1}$, v} induce the graph G_1^n, $n \geq 2$ from Figure 1.

Conversely, suppose $G_P \in \mathcal{F}(G_1^n)$. Then G_P contains an induced subgraph G_1^n, $n \geq 2$ shown in Figure 1(b), with vertices labeled by u, a_0, $a_1, a_2, a_3, \ldots, a_{2n-1}$, v. The set of vertices **{**$a_2, a_4, \ldots, a_{2n}$} is an independent set in G_1^n, $n \geq 2$. Therefore these vertices lie on a common chain in P (by Lemma 1*(ii)*) and they are not in a covering relation. Denote the chain containing {$a_2, a_4, \ldots, a_{2n}$} by C. Then the following cases (i) and (ii) cannot occur.

(i): $a_{2i-4} \lhd\lhd a_{2i} \lhd\lhd a_{2i-2}, i \geq 3$:

Since a_{2i-3} and a_{2i} are nonadjacent in G they lie on a common chain in P.

$a_{2i-3} \lhd\lhd a_{2i}$: then $a_{2i-3} \lhd\lhd a_{2i} \lhd\lhd a_{2i-2}$ in P, contradicting a_{2i-3} and a_{2i-2} are adjacent in G.

$a_{2i} \lhd\lhd a_{2i-3}$: then $a_{2i-4} \lhd\lhd a_{2i} \lhd\lhd a_{2i-3}$ in P, contradicting a_{2i-4} and a_{2i-3} are adjacent in G.

The same contradiction arise if $a_{2i-2} \lhd\lhd a_{2i} \lhd\lhd a_{2i-4}$.
(ii): $a_{2i-2} \lhd\lhd a_{2i-4} \lhd\lhd a_{2i}$, $i \geq 3$:
Since a_{2i-4} and a_{2i-1} are nonadjacent in G, they lie on a common chain in P.
$a_{2i-4} \lhd\lhd a_{2i-1}$: then $a_{2i-2} \lhd\lhd a_{2i-4} \lhd\lhd a_{2i-1}$ in P, contradicting a_{2i-2} and a_{2i-1} are adjacent in G.
$a_{2i-1} \lhd\lhd a_{2i-4}$: then $a_{2i-1} \lhd\lhd a_{2i-4} \lhd\lhd a_{2i}$ in P, contradicting a_{2i-1} and a_{2i} are adjacent in G.
The same contradiction arise if $a_{2i} \lhd\lhd a_{2i-4} \lhd\lhd a_{2i-2}$, $i \geq 3$.
The only possible cases are $a_{2i-4} \lhd\lhd a_{2i-2} \lhd\lhd a_{2i}$, $i \geq 3$and $a_{2i} \lhd\lhd a_{2i-2} \lhd\lhd a_{2i-4}$, $i \geq 3$. Without loss of generality, assume that $a_{2i-4} \lhd\lhd a_{2i-2} \lhd\lhd a_{2i}$. Since a_{2i-3} is adjacent to $a_{2i-4}$and a_{2i-2} and a_{2i-1} is adjacent to a_{2i-2} and a_{2i} in G, we have $a_2 \lhd a_3 \lhd a_4 \lhd \ldots \lhd a_{2n}$. Again a_1 is adjacent to $a_2$and nonadjacent to all vertices of G in C, we have $a_1 \lhd a_2$. Also u is adjacent to a_1 and v is adjacent to a_{2n}in G and these vertices are nonadjacent to all other vertices in G implies $u \lhd a_1 \lhd a_2 \lhd a_3 \lhd a_4 \lhd \ldots \lhd a_{2n} \lhd v$; then we have two possibilities.

Case (1): $a_1 \lhd a_0$ in P: again we have two possibilities.
Subcase (1.1) $a_0 \lhd a_{2n}$ in P: take (a_1, a_0) and (a_0, a_{2n}) as normal edges and avoid all dashed lines in Figure 2 to obtain the $\lhd$ - preserving subposet P_1in Figure 3.

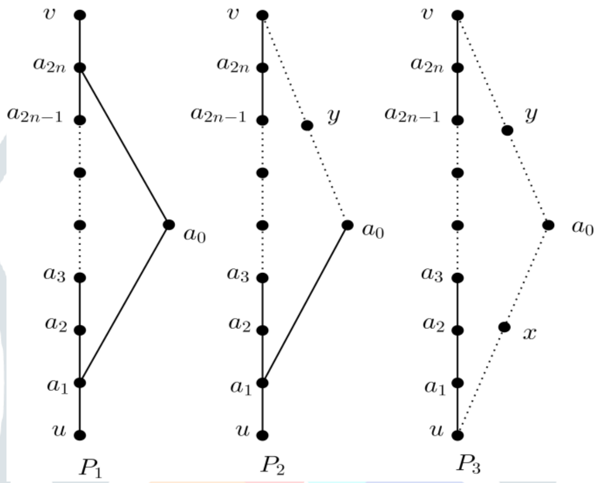

Figure 3: $\lhd$ - preserving subposets corresponding to Q_1.

Subcase (1.2) $a_{2n} \| a_0$ in P: take the chain from v to a_0 through y to obtain the $\lhd$- preserving subposet P_2in Figure 3.
Case (2): $a_1 \| a_0$in P: take the chain from u to a_0 through x to obtain the $\lhd$- preserving subposet P_3in Figure 3.
All posets in Figure 3 are represented by a single 3-colored diagram Q_1 in Figure 2. □

Since Figure 1(a) has an induced cycle of length 5, it is not an induced subgraph of Cover-incomparable graph(by Lemma 3.3,[2]).

Remarks
Here we characterize forbidden $\lhd$ - preserving subposets of G_1^n in Figure1 by Theorem 2 and introduce the idea of 3-colored diagrams to minimize the list of subposets.

References

[1] B. Brešar, M. Changat, T. Gologranc, J. Mathew, A. Mathews, Coverincomparability graphs and chordal graphs, Discrete Appl.
Math. 158 (2010) 1752-1759.
[2] B. Brešar, M. Changat, S. Klavžar, M. Kovše, J. Mathew, A. Mathews, Coverincomparability graphs of posets, Order 25 (2008)
335- 347.
[3] B. Brešar, M. Changat, S. Klavžar, J. Mathew, A. Mathews, Prasanth G. N., Characterizing posets for which their natural transit
functions coincide, Ars Math. Contemp. 2 (2009) 27-33.
[4] M. Changat, S. Klavžar and H. M. Mulder, The all-paths transit function of a graph, Czech. Math. J. 51(126) (2001) 439-448.
[5] M. Changat and J. Mathews, Induced path transit function, monotone and Peano axioms, Discrete Math. 286 (2004) 185-194.
[6] A. Mathews and J. Mathews, Transit functions on posets and lattices. In: Convexity in Discrete Structures (M. Changat, S. Klavžar,H.M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 105-116.
[7] A. Brandstädt, V. B. Le, J. P. Spinrad: Graph Classes: A Survey, SIAM Monographs on Discrete Mathematics and Applications,
3, 1999.
[8] J.Book , J. Maxová, Characterizing subclasses of cover-incomparability graphs by forbidden subposets,Order manuscript.
[9] J. Maxová, P. Pavlìková, D. Turzìk, On the complexity of cover-incomparability graphs of posets, Order 26 (2009), 229-236.
[10] J. Maxová, D. Turzìk, Which distance-hereditary graphs are cover-incomparability graphs?, Discrete Appl. Math.161(2013),

2095-2100.
[11] H. M. Mulder, Transit functions on graphs (and posets). In: Convexity in Discrete Structures (M. Changat, S. Klav_zar, H.M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 117-130.
[12] B. Brešar, M. Changat, Tanja Glorance,M. Kov_se, Baiju Sukumaran, Cover- incomparability graphs and 2-colored posets, Taiwanese Journal of Mathematics, DOI: 10.11650/tjm.18.2014.3697
[13] Baiju Sukumaran "2-Colored and 3-Colored Diagrams of Posets in Cover-incomparability Graphs", International Journal of Mathematics Trends and Technology (IJMTT). V53(4):267-269 January 2018. ISSN:2231-5373.
[14] Gabriele Di Stefano, Distance- hereditary comparability graphs, Discrete Applied Mathematics 160(2012) 2669 2680.