Laplacian Minimum Dominating Energy of some special classes of Graphs.

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Abstract:
In this paper, we defined set energy of a graph. We also study the special case of a set S and the corresponding Laplacian minimum dominating energy of S for some special classes of graphs. We also attained their bounds.

Keywords:
Minimum dominating set, Laplacian minimum dominating matrix, Laplacian minimum dominating eigen values, Laplacian minimum dominating energy of a graph.

Subject classification: 05C50, 05C69.

1. Introduction:
The concept of energy of a graph was introduced by I.Gutman [8] in 1978.

Let G=(V,E) be a simple, finite, connected undirected graph with order n and size m. A dominating set in G is a subset D of V(G) such that each element of V(G) –D is adjacent to at least one vertex of D by means of a matrix as follows; in the adjacency matrix A(G) of G replace the a_{ii} by 1 if and only if Vi \epsilon S and A= (a_{ij}) be the adjacency matrix of the graph G and the eigen values of the adjacency matrix are \lambda_1, \lambda_2, ... \lambda_n. It is assumed that these eigen values are in the non increasing order.

The Energy E(G) of a graph G is defined to be the sum of the absolute values of the eigen values of G, i.e, E(G) = \sum_{i=1}^{n} |\lambda_i| . For the details on the mathematical aspects of the theory of graph energy, we can make reviews [11] and the references cited therein.

I.Gutman and B.Zhou [2] defined the Laplacian energy of a graph G in the year 2006. The Laplacian matrix of the graph G denoted by L=(L_{ij}) is a square matrix of order n whose elements are defined as

L_{ij} = \begin{cases} 
-1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\
0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\
d_i & \text{if } i = j, \text{where } d_i \text{ is the degree of the vertex} 
\end{cases}

Let \mu_1, \mu_2, ...........\mu_n be the Laplacian eigen values of G. Laplacian energy L[E(G)] of G is defined as L[E(G)] = \sum_{i=1}^{n} \mu_i - \frac{2m}{n}. The basic properties including various upper and lower bounds for Laplacian energy of a graph G have been established in [5,6,7,13,14,19,20,21] and it has found remarkable chemical applications, the molecular orbital theory of conjugated molecules [7].
Definition 1.1:

The minimum Dominating Energy of a graph G:

Let $G=(V,E)$ be a simple graph of order $n$ with vertex set $V=\{v_1,v_2,\ldots,v_n\}$ and size $m$. A non-empty subset $D$ of $V$ is called a dominating set of $G$ if every vertex in $V-D$ is adjacent to at least one vertex in $D$. Any Dominating set with minimum cardinality is called the minimum dominating set.

Let $D$ be the minimum dominating set of a graph $G$. The adjacency matrix of the minimum dominating set of $G$ is the $(n\times n)$ matrix denoted by $A_D(G)$ and it is defined as $A_D(G) = (a_{ij})$ where

\[
a_{ij} = \begin{cases} 
1 & \text{if } v_i, v_j \in D \\
1 & \text{if } i = j \text{ and } v_i \in D \\
0 & \text{otherwise}.
\end{cases}
\]

The characteristic polynomial of $A_D(G)$ is denoted by $f_n(G,\lambda) = \det(\lambda I - A_D(G))$. The minimum dominating eigen values of the graph $G$ are the eigen values of $A_D(G)$. Since $A_D(G)$ is real and symmetric matrix its eigen values are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

The minimum dominating energy of $G$ is defined as

\[
E_D(G) = \sum_{i=1}^{n} |\lambda_i| \quad \text{and the trace of } A_D(G) = \text{Domination number of the graph } G.
\]

Definition 1.2:

Laplacian minimum Dominating energy of a graph $G$:

Let $D(G)$ be the diagonal matrix denoting the vertex degrees of the adjacent vertices of the graph $G$. The Laplacian minimum dominating matrix of $G$ is denoted by $L_D(G)$ and it is defined as $L_D(G) = D(G) - A_D(G)$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigen values of $L_D(G)$, arranged in non-increasing order. These eigen values are called Laplacian minimum dominating eigen values of $G$.

The Laplacian minimum dominating energy of the graph $G$ is defined as

\[
LE_D(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}| \quad \text{where } m \text{ and } n \text{ are the order and the size of the graph } G \text{ and } \frac{2m}{n} \text{ denotes the average degree of } G.
\]

Laplacian Minimum Dominating Energy of some special classes of graph.

Definition 2.1:

Friendship graph or Dutch Windmill graph:

Friendship graph is a planar undirected graph with order $2n+1$ and size $3n$. It states that the finite graphs with the property that every two vertices have exactly one neighbour in common.
Theorem 2.1:

For $n \geq 2$, the Laplacian minimum dominating energy of friendship graph $k_{1,n}$ or $n$-fan graph is approximately.

Proof:
Consider the friendship graph $k_{1,n}$ with vertex $V=\{1,2,3,\ldots,n\}$ with centre at 1. Then minimum dominating set is $D=\{1\}$ and hence the domination number $\gamma(G)=1$.

Consider the adjacency matrix of the minimum dominating matrix of $k_{1,n}$ is

$$A_D(K_{1,n}) = \begin{pmatrix} 1 & 1 & 1 & \ldots \ldots & 1 \\ 1 & 1 & 1 & \ldots \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots \ldots & 1 \end{pmatrix}_{(2n+1) \times (2n+1)}$$

and the diagonal matrix of $K_{1,n}$ is

$$D(K_{1,n}) = \begin{pmatrix} 2n & 0 & 0 & \ldots \ldots & 0 \\ 0 & 2 & 0 & \ldots \ldots & 0 \\ 0 & 0 & 2 & \ldots \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots \ldots & 2 \end{pmatrix}_{(2n+1) \times (2n+1)}$$

Theorem 2.1 states that for $n \geq 2$, the Laplacian minimum dominating energy of friendship graph $k_{1,n}$ or $n$-fan graph is approximately $n(2n+1) / 2n+1$. This is because the minimum dominating set is $D=\{1\}$, and the Laplacian matrix $L(D)$ is given by $L(D)=D(D)D^{-1}$, where $D=D(K_{1,n})$. The Laplacian minimum dominating energy is then given by the expression $\text{tr}(L(D)) = n(2n+1) / 2n+1$. This result is based on the properties of the adjacency and diagonal matrices of the friendship graph $k_{1,n}$. The proof involves considering the specific structure of the friendship graph and applying the properties of diagonal and adjacency matrices to derive the result.
The Laplacian minimum dominating matrix of $K_{1,n}$ is given by

$$L_D(K_{1,n}) = D(K_{1,n}) - A_D(K_{1,n}).$$

$$L_D(K_{1,n}) = \begin{pmatrix}
2n-1 & -1 & -1 & \ldots & -1 \\
-1 & 1 & -1 & \ldots & 0 \\
-1 & -1 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & 1
\end{pmatrix} (2n+1) \times (2n+1)$$

The characteristic equation of $L_D(K_{1,n})$ is $\mu (\mu - 1)^{n-1}(\mu - 3)^n(\mu - 2n) = 0$

Average degree of the Friendship graph $K_{1,n} = \frac{6n}{2(n+1)}$

Laplacian minimum dominating energy of

$$LE_D(K_{1,n}) = |0 - \frac{6n}{2(n+1)}| + |1 - \frac{6n}{2(n+1)}| (n-1) + |3 - \frac{6n}{2(n+1)}| n + |2n - \frac{6n}{2(n+1)}|$$

$$= \frac{6n}{2(n+1)} + \frac{2(n+1)-6n}{2(n+1)} (n-1) + \frac{3(2n+1)-6n}{2(n+1)} n + \frac{2n(2n+1)-6n}{2(n+1)}$$

$$= \frac{n(8n+1)}{2(n+1)}.$$

**Definition 2.2:**

**Wheel Graph**

Wheel graphs are planar graphs. A wheel graph $W_n$ is a graph with order $n$ and size $2(n-1)$ ($n \geq 4$) formed by connecting a single vertex to all the vertices of a $(n-1)$ cycle.

**Theorem 2.2:**

For $n \geq 2$, the Laplacian minimum dominating energy of the Wheel graph $W_{1,n-1}$ is $\frac{2(n^2-5n+2)}{n}$ approximately.

**Proof:**

Consider the wheel graph $W_n$ with vertex set $\{v_1, v_2, v_3, \ldots, v_n\}$. The minimum dominating set is $D = \{v_1\}$ and hence the domination number $\gamma(G) = 1$. Consider the adjacency matrix of the minimum dominating set of $W_{1,n-1}$.
The diagonal matrix of \( W_{1,n-1} \) is

\[
D(W_{1,n-1}) = \begin{pmatrix}
n-1 & 0 & 0 & \cdots & 0 \\
0 & 3 & 0 & \cdots & 0 \\
0 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 3
\end{pmatrix}
\]

(nXn)

The Laplacian minimum dominating matrix is given by

\[
L_D(W_{1,n-1}) = D(W_{1,n-1}) - A(W_{1,n-1})
\]

\[
L_D(W_{1,n-1}) = \begin{pmatrix}
n - 2 & -1 & -1 & \cdots & -1 \\
-1 & 2 & -1 & \cdots & -1 \\
-1 & -1 & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & 0 & \cdots & 2
\end{pmatrix}
\]

(nXn)

The characteristic equation of \( L_D(W_{1,n-1}) \) is

\[
\mu (\mu - 2) \frac{n-3}{2} (\mu - 4) \frac{n-3}{2} (\mu - n + 2)(\mu - n + 1) = 0
\]

Average degree of the wheel graph \( W_{1,n-1} \) is

\[
\frac{4(n-1)}{n}
\]

Laplacian minimum dominating energy of \( L_D(W_{1,n-1}) \) is

\[
LE_D(W_{1,n-1}) = \frac{4(n-1)}{n} - \frac{(n-4)(n-3)}{2n} + \frac{4(n-3)}{n} \left( \frac{n-3}{2} \right) + \frac{n-2}{n} - \frac{(n-1)}{n}
\]

\[
\geq \frac{2(n^2-5n+2)}{n}
\]
3. Upper bounds of Laplacian minimum dominating energy of Friendship and Wheel graph

Theorem:

Let G be a graph with order n and size m and D is the minimum dominating set of G then

$$\text{LE}_D(G) \leq \sqrt{2Mn + 4m(1 - m)}$$

Proof:

Cauchy–Schwarz inequality is

$$\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right)$$

Put $$a_i = 1$$, $$b_i = \left| \mu_i - \frac{2m}{n} \right|$$ then

$$\left( \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|^2 \right) \leq \left( \sum_{i=1}^{n} 1 \right) \left( \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|^2 \right)$$

i.e., $$[\text{LE}_D(G)]^2 = n \left[ \sum_{i=1}^{n} \mu_i^2 + \sum_{i=1}^{n} \frac{4m^2}{n^2} - \frac{4m}{n} \sum_{i=1}^{n} \mu_i \right]$$

$$= n \left[ 2M + \frac{4m^2}{n^2} - \frac{4m}{n} \left( 2m - |D| \right) \right]$$

$$= n \left[ 2M + \frac{4m^2}{n} - \frac{8m^2}{n} + \frac{4m |D|}{n} \right]$$

$$= 2Mn + 4m ( |D| - m )$$

$$\text{LE}_D(G) \leq \sqrt{2Mn + 4m( |D| - m )}$$

In both Friendship and Wheel graph, the cardinality of minimum dominating set is one.

i.e., $$|D| = 1$$

Hence $$\text{LE}_D(G) \leq \sqrt{2Mn + 4m(1 - m)}$$.

References:


