Fuzzy Strongly $\alpha$-Continuous Maps on generalized Topological Spaces

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Abstract: In this paper we have studied g-fuzzy strongly $\alpha$-continuous maps and have investigated equivalent conditions for a map to be g-fuzzy strongly $\alpha$-continuous map. Also we have established some significant properties of g-fuzzy strongly $\alpha$-continuous maps.

IndexTerms - g-fuzzy topology, g-fuzzy semi-open, g-fuzzy $\alpha$-open, g-fuzzy continuous, g-fuzzy semi-continuous, g-fuzzy $\alpha$-continuous.

1. INTRODUCTION

Csaszar introduced the notions of generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. Moreover, he studied the simplest separation $Y$. Beceren has introduced the concept of strongly $\alpha$-continuous maps in 2000. Bin Shahana has defined the concept of fuzzy $\alpha$-continuous maps. Fuzzy strongly $\alpha$-continuous map is the stronger form of fuzzy $\alpha$-continuous map. In the present paper, we further carry out our investigations on generalized open fuzzy sets and generalized fuzzy topology. In this paper we have studied g-fuzzy strongly $\alpha$-continuous maps and investigated necessary and sufficient conditions for a map to be g-fuzzy strongly $\alpha$-continuous. Further we have established some important properties of g-fuzzy strongly $\alpha$-continuous maps.

2. Preliminaries.

Definition 2.1: Let $X$ be a crisp set and let $\tau_g$ be a collection of fuzzy sets on $X$. Then $\tau_g$ is said to be a generalized fuzzy (g-fuzzy) topological space on $X$ if it satisfies following conditions:

i) The fuzzy sets 0 and 1 are in $\tau_g$ where $0, 1: X \to I$ are defined as $0(x) = 0$ and $1(x) = 1 \quad \forall \ x \in X$

ii) If $\{\lambda_j\}, j \in J$ is any family of fuzzy sets on $X$ where $\lambda_j \in \tau_g$ then $\cup_{j \in J} \lambda_j \in \tau_g \quad \forall \ j \in J$.

The pair $(X, \tau_g)$ is called a generalized fuzzy topological space. We usually denote fuzzy topological space $(X, \tau_g)$ by $X_g$ only. The members of the collection $\tau_g$ are called g-fuzzy open sets. In $X_g$ a fuzzy set $\lambda: X_g \to I$ is called a fuzzy closed set in $X_g$ provided its complement $\lambda^c$ is a fuzzy open set in $X_g$. In a generalized fuzzy topological space $X_g$, the closure of a fuzzy set denoted by Cl($\lambda$) is defined to be the intersection of all fuzzy closed sets in $X_g$ containing $\lambda$. The interior of a fuzzy set $\lambda$ is denoted by Int($\lambda$) and is defined to be the union of all fuzzy open sets in $X_g$ contained in $\lambda$. Clearly Cl($\lambda$) is a fuzzy closed set in $X_g$ and $\text{Int}(\lambda)$ is a fuzzy open set in $X_g$. We note that Cl($\lambda^c$) = Int($\lambda$)$^c$ and $\text{Int}(\lambda^c)$ = Cl($\lambda$)$^c$. If $\lambda_j: X_g \to I, \ j \in J$ is any arbitrary collection of fuzzy sets in $X_g$ then $\text{Int}(\bigcup_{j \in J} \lambda_j) \supseteq \bigcup_{j \in J} \text{Int}(\lambda_j)$ and $\text{Cl}(\bigcup_{j \in J} \lambda_j) \supseteq \bigcup_{j \in J} \text{Cl}(\lambda_j)$.

Definition 2.2: Let $(X, \tau_g)$ be a generalized fuzzy topological space. A fuzzy set $\lambda$ in $X$ is called:

i) g-fuzzy semi-open if $\lambda \subseteq \text{Cl}(\text{Int}(\lambda))$.

ii) g-fuzzy pre-open if $\lambda \subseteq \text{Int}(\text{Cl}(\lambda))$.

iii) g-fuzzy $\alpha$-open if $\lambda \subseteq \text{Int}(\text{Cl}(\lambda))$.

iv) g-fuzzy semipro-open if $\lambda \subseteq \text{Cl}(\text{Int}(\text{Cl}(\lambda)))$.

Remark 2.1: Every fuzzy open set is g-fuzzy $\alpha$-open, every g-fuzzy $\alpha$-open set is g-fuzzy semi open (resp. fuzzy pre-open) and every g-fuzzy semi open (resp. Fuzzy pre-open) set is g-fuzzy semi pre-open. But the converses may not be true. The complement
of a g-fuzzy semi-open (resp. fuzzy pre-open, fuzzy α-open, fuzzy semi pre-open) set is called g-fuzzy semi-closed (resp. fuzzy pre-closed, fuzzy α-closed, fuzzy semi pre-closed).

**Definition 2.3:** Let \((X, \tau_g)\) be g-fuzzy topological space. A fuzzy set \(X_r : X \to \Gamma\) where \(x \in X\) and \(0 < r \leq 1\) is said to be a g-fuzzy point in a g-fuzzy topological space \(X\) if

\[
X_r(y) = \begin{cases} r, & \text{if } y = x \\ 0, & \text{if } y \neq x, \quad y \in X \end{cases}
\]

The g-fuzzy point \(X_r\) belongs to a g-fuzzy set \(1\) in \(X\) if \(r \leq \lambda(x)\) i.e. \(X_r\) is a subset of \(1\).

**Definition 2.4:** Let \((X, \tau_g)\) be a generalized fuzzy topological space. A sub-collection \(B\) of \(X\) is called a base for \(\tau_g\) if each member of \(\tau_g\) can be expressed as the union of members of \(B\).

**Definition 2.5:** Let \(X\) and \(Y\) be generalized fuzzy topological spaces. If \(\lambda\) is a fuzzy set in \(X\) and \(\mu\) is a fuzzy set in \(Y\) then \(\lambda \times \mu\) is a g-fuzzy set in \(X \times Y\) defined as \((\lambda \times \mu)(x, y) = \min(\lambda(x)\mu(y)), \forall x \in X, y \in Y\). If \(\lambda\) is a g-fuzzy open set in \(X\) and \(\mu\) is a g-fuzzy open set in \(Y\) then \(\lambda \times \mu\) is called a g-fuzzy open set in \(X \times Y\). A g-fuzzy open set in \(X \times Y\) is arbitrary union of g-fuzzy basic open sets in \(X \times Y\). The collection \(\tau_g\) of all g-fuzzy open sets in \(X \times Y\) is called g-fuzzy product topology on \(X \times Y\) and \(X \times Y\) together with \(\tau_g\) is called the generalized fuzzy product topological space of \(X\) and \(Y\).

### 3. Fuzzy Strongly α-Continuous Maps

**Definition 3.1** Let \(X\) and \(Y\) be generalized fuzzy topological spaces and \(f : X \to Y\) be a map. Then \(f\) is said to be:

i) g-fuzzy continuous if \(f^{-1}(\lambda)\) is g-fuzzy open in \(X\) for each g-fuzzy open set \(\lambda\) in \(Y\).

ii) g-fuzzy α-continuous if \(f^{-1}(\lambda)\) is g-fuzzy α-open in \(X\) for each g-fuzzy open set \(\lambda\) in \(Y\).

iii) g-fuzzy semi-continuous if \(f^{-1}(\lambda)\) is g-fuzzy semi-open in \(X\) for each g-fuzzy open set \(\lambda\) in \(Y\).

iv) g-fuzzy strongly α-continuous if \(f^{-1}(\lambda)\) is g-fuzzy α-open in \(X\) for each g-fuzzy semi-open set \(\lambda\) in \(Y\).

**Example 3.1:** Let \(X = \{x_1, x_2\}\) and \(\mu\) and \(\lambda\) be g-fuzzy sets in \(X\), \(Y\) defined as, \(\mu = \{(x_1, 0.5), (x_2, 0.6)\}\) and \(\lambda = \{(y_1, 0.7), (y_2, 0.8)\}\). Let \(\tau_g = \{0, \mu, 1\}\) and \(\tau_g' = \{0, \lambda, 1\}\) be generalized fuzzy topologies on sets \(X\) and \(Y\) respectively. The map \(f : X \to Y\) defined as \(f(x_1) = y_1, f(x_2) = y_2\) is g-fuzzy strongly α-continuous map. Since each g-fuzzy open set is g-fuzzy semi-open, it follows that every g-fuzzy strongly α-continuous map is g-fuzzy α-continuous. However a g-fuzzy α-continuous map may not be g-fuzzy strongly α-continuous. We have the following example.

**Example 3.2:** Let \(X = \{x_1, x_2\}\) and \(\mu, \lambda\) and \(\nu\) be fuzzy sets in \(X\) defined as \(\mu = \{(x_1, 0.2), (x_2, 0.3)\}\), \(\lambda = \{(y_1, 0.5), (y_2, 0.6)\}\) and \(\nu = \{(y_1, 0.7), (y_2, 0.7)\}\). Let \(Y = \{y_1, y_2\}\) and \(\gamma\) be g-fuzzy set in \(Y\) defined as \(\gamma = \{(y_1, 0.5), (y_2, 0.7)\}\). Consider the generalized fuzzy topologies \(\tau_g = \{0, \mu, \lambda, 1\}\) and \(\tau_g' = \{0, \nu, 1\}\) on sets \(X\) and \(Y\) respectively. The map \(f : X \to Y\) defined as \(f(x_1) = y_1, f(x_2) = 1, 2\) is g-fuzzy α-continuous map. The fuzzy set \(\eta\) in \(Y\) defined as \(\eta = \{(y_1, 0.7), (y_2, 0.8)\}\) is fuzzy semi-open set in \(Y\) but \(f^{-1}(\eta)\) is not g-fuzzy α-open set in \(X\). Hence the map \(f : X \to Y\) is g-fuzzy α-continuous but not g-fuzzy strongly α-continuous. The concepts of α-contiguity and g-fuzzy strongly α-contiguity are independent of each other. The map \(f : X \to Y\) mentioned in Example 3.1 is g-fuzzy strongly α-continuous but not fuzzy continuous. In the following example we see that a g-fuzzy continuous map \(f : X \to Y\) may not be g-fuzzy strongly α-continuous.

**Example 3.3** Let \(X = \{x_1, x_2\}\) and \(Y = \{y_1, y_2\}\). Let \(\mu, \lambda\) and \(\nu\) be fuzzy sets in \(X\) and \(\gamma\) be fuzzy set in \(Y\) defined as \(\mu = \{(x_1, 0.3), (x_2, 0.4)\}\), \(\nu = \{(y_1, 0.5), (y_2, 0.6)\}\) and \(\tau_g = \{0, \mu, \lambda, 1\}\) and \(\tau_g' = \{0, \nu, 1\}\) be generalized fuzzy topologies on sets \(X\) and \(Y\) respectively. The map \(f : X \to Y\) defined as \(f(x_1) = y_2, f(x_2) = 1, 2\) is g-fuzzy continuous. The fuzzy set \(\eta\) in \(Y\) defined as \(\eta = \{(y_1, 0.6), (y_2, 0.7)\}\) is fuzzy semi-open set in \(Y\) but \(f^{-1}(\eta)\) is not g-fuzzy α-open set in \(X\). Hence the map \(f : X \to Y\) is g-
fuzzy continuous but not g-fuzzy strongly α-continuous. In the following result we have obtained several equivalent conditions for a map \( f: X \rightarrow Y \), where \( X \) and \( Y \) are generalized fuzzy topological spaces to be g-fuzzy strongly \( \alpha \)-continuous.

**Theorem 3.1** Let \( X \) and \( Y \) be generalized fuzzy topological spaces and \( f: X \rightarrow Y \) be a map. Then following conditions are equivalent:

i) \( f \) is g-fuzzy strongly \( \alpha \)-continuous.

ii) For each g-fuzzy point \( x_\beta \) in \( X \) and each g-fuzzy semi-open set \( \lambda \) in \( Y \) containing \( f(x_\beta) \), there exists a g-fuzzy \( \alpha \)-open set \( \mu \) in \( X \) containing \( x_\beta \) such that \( h(\mu) \leq \lambda \).

iii) For each g-fuzzy semi-closed set \( \lambda \) in \( Y \), \( f^{-1}(\lambda) \) is g-fuzzy closed set in \( X \).

iv) For each g-fuzzy set \( \mu \) in \( X \), \( f(\alpha Cl(\mu)) \leq sCl(f(\mu)) \).

v) For each g-fuzzy set \( \lambda \) in \( Y \), \( \alpha Cl(f^{-1}(\mu)) \leq f^{-1}(sCl(\mu)) \).

vi) For each g-fuzzy set \( \lambda \) in \( Y \), \( Cl(\text{Int}(f^{-1}(\lambda))) \leq f^{-1}(sCl(\lambda)) \).

vii) For each g-fuzzy set \( \mu \) in \( X \), \( f(Cl(\text{Int}(\mu)))) \leq sCl(f(\mu)) \).

**Proof** (i) \( \Rightarrow \) (ii): Let \( f: X \rightarrow Y \) be a g-fuzzy strongly \( \alpha \)-continuous map. Let \( x \) where \( x_\beta \in X \) and \( 0 < \beta \leq 1 \) be a g-fuzzy point in \( X \) and \( \lambda \) be a g-fuzzy semi-open set in \( Y \) containing the g-fuzzy point \( f(x_\beta) \). Since \( f(x_\beta)(f(x)) = \beta = \lambda(f(x)) \), we have \( \beta \leq \lambda\( f(x)) \) i.e. \( f^1(\lambda) \) contains the g-fuzzy point \( x_\beta \). Since \( f \) is g-fuzzy strongly \( \alpha \)-continuous, \( \mu \) is g-fuzzy \( \alpha \)-open set in \( X \) containing the g-fuzzy point \( x_\beta \) and \( f(\mu) \leq \lambda \).

(ii) \( \Rightarrow \) (i): Let \( \lambda \) be a g-fuzzy semi-open set in \( Y \). For \( x \in X \) and \( 0 < \beta \leq 1 \). Let \( x_\beta \) be a g-fuzzy point in \( f^{-1}(\lambda) \). Then \( \lambda \) contains \( f(x_\beta) \) and so by given condition (ii) there exists a g-fuzzy \( \alpha \)-open set \( \mu \) in \( X \) containing the g-fuzzy point \( x_\beta \) and \( f(\mu) \leq \lambda \). This implies \( \mu \leq \text{Int}(\text{Cl}(\text{Int}(\mu))) \leq \text{Int}(\text{Cl}(f^{-1}(\lambda))) \) and therefore \( \text{Int}(\text{Cl}(f^{-1}(\lambda))) \) contains the g-fuzzy point \( x_\beta \). Thus each g-fuzzy point of \( f^1(\lambda) \) is also a g-fuzzy point of \( \text{Int}(\text{Cl}(f^{-1}(\lambda))) \). This shows that in \( f^{-1}(\lambda) \leq \text{Int}(\text{Cl}(f^{-1}(\lambda))) \) i.e., \( f^{-1}(\lambda) \) is a g-fuzzy \( \alpha \)-open set in \( X \). Hence \( f: X \rightarrow Y \) is g-fuzzy strongly \( \alpha \)-continuous map.

(i) \( \Rightarrow \) (iii): Let \( \lambda \) be a g-fuzzy semi-open set in \( Y \). Then \( \lambda = 1 - \lambda \) is g-fuzzy semi-closed set in \( Y \). Since \( f: X \rightarrow Y \) is g-fuzzy strongly \( \alpha \)-continuous, \( f^{-1}(\lambda) = f^{-1}(1 - \lambda) \) is g-fuzzy \( \alpha \)-open set in \( X \). This implies \( f^{-1}(\lambda) = f^{-1}(1 - \lambda) = 1 - f^{-1}(\lambda^c) \) is g-fuzzy \( \alpha \)-closed set in \( X \).

(iii) \( \Rightarrow \) (i): Let \( \lambda \) be a fuzzy semi-closed set in \( Y \). Then \( \lambda^c = 1 - \lambda \) is fuzzy semi-open set in \( Y \). Therefore by given condition (iii), \( f^{-1}(\lambda^c) = f^{-1}(1 - \lambda) \) is g-fuzzy \( \alpha \)-closed set in \( X \). Hence \( f^{-1}(\lambda) = f^{-1}(1 - \lambda^c) = 1 - f^{-1}(\lambda^c) \) is g-fuzzy \( \alpha \)-open set in \( X \). Thus \( f: X \rightarrow Y \) is g-fuzzy strongly \( \alpha \)-continuous map.

(iii) \( \Rightarrow \) (iv): Let \( \mu \) be a g-fuzzy set in \( X \). Since \( \mu \leq f^{-1}(f(\mu)) \) we have \( \mu \leq f^{-1}(sCl(\mu)) \). Now \( sCl(f(\mu)) \) is a g-fuzzy semi-closed set in \( Y \). Hence by given condition (iii), \( f^{-1}(sCl(\mu)) \) is g-fuzzy \( \alpha \)-closed set in \( X \). Since \( sCl(f(\mu)) \) is the smallest g-fuzzy \( \alpha \)-closed set containing \( \mu \), it follows that \( sCl(f(\mu)) \leq f^{-1}(sCl(\mu)) \). This implies \( f(\alpha Cl(\mu)) \leq sCl(f(\mu)) \) and hence \( f(\alpha Cl(\mu)) = sCl(f(\mu)) \).

(iv) \( \Rightarrow \) (iii): Let \( \lambda \) be a g-fuzzy semi-closed set in \( X \). Then by given condition (iv), we have \( f^{-1}(\alpha Cl(f^{-1}(\lambda))) \leq sCl(f(f^{-1}(\lambda))) \). This implies \( f^{-1}(\alpha Cl(f^{-1}(\lambda))) \leq f^{-1}(\lambda) \). Since \( f^{-1}(\lambda) \leq \alpha Cl(f^{-1}(\lambda)) \), we deduce that \( f^{-1}(\lambda) = \alpha Cl(f^{-1}(\lambda)) \). Now \( \alpha Cl(f^{-1}(\lambda)) \) is g-fuzzy \( \alpha \)-closed set in \( X \), it follows that \( f^{-1}(\lambda) \) is g-fuzzy \( \alpha \)-closed set in \( X \).

(iv) \( \Rightarrow \) (v): Let \( \mu \) be a g-fuzzy set in \( X \). Then by given condition (iv), \( f(\alpha Cl(f^{-1}(\lambda))) \leq sCl(f(f^{-1}(\lambda))) = sCl(f(\mu)) \). This implies \( \alpha Cl(f^{-1}(\lambda)) \leq f^{-1}(sCl(\mu)) \), and hence \( f(\alpha Cl(\mu)) \leq sCl(f(\mu)) \).

(i) \( \Rightarrow \) (vi): Let \( \lambda \) be a g-fuzzy set in \( Y \). Since \( sInt(\lambda) \) is a g-fuzzy semi-open set in \( Y \), by given condition (i), \( f^{-1}(sInt(\lambda)) \) is g-fuzzy \( \alpha \)-open set in \( X \). Hence we have \( f^{-1}(sInt(\lambda)) = sInt(f^{-1}(sInt(\lambda))) \). Since \( sInt(f^{-1}(sInt(\lambda))) \leq sInt(f^{-1}(\lambda)) \), we find that \( f^{-1}(sInt(\lambda)) \leq sInt(f^{-1}(\lambda)) \).
Let $\lambda$ be a g-fuzzy semi-open set in $Y$. Then we have $s\text{Int}(\lambda) = \lambda$. Therefore by given condition (vi) $f^i(\lambda) = f^i(s\text{Int}(\lambda)) \subseteq \alpha\text{Int}(f^i(\lambda))$, i.e. $f^i(\lambda) \subseteq \alpha\text{Int}(f^i(\lambda))$. Since $\alpha\text{Int}(f^i(\lambda)) \subseteq f^i(\lambda)$, we find that $f^i(\lambda) = \alpha\text{Int}(f^i(\lambda))$. Hence $f^i(\lambda)$ is a g-fuzzy $\alpha$-open set in $X$. Thus $f : X \to Y$ is a g-fuzzy strongly $\alpha$-continuous map.

**Theorem 3.2:** Let $X$ and $Y$ be generalized fuzzy topological spaces and $f : X \to Y$ be a bijective map. Then $f$ is g-fuzzy strongly $\alpha$-continuous iff for each g-fuzzy set $\mu$ in $X$, $s\text{Int}(f(\mu)) \leq f(\alpha\text{Int}(\mu))$.

**Proof:** Let $f : X \to Y$ be a bijective map. Suppose $f$ is g-fuzzy strongly $\alpha$-continuous. If $\lambda$ is a g-fuzzy set in $X$ then $f(\lambda)$ is a g-fuzzy set in $Y$. Since $f$ is g-fuzzy strongly $\alpha$-continuous, from Theorem 3.6 we have, $f^i(s\text{Int}(f(\mu))) \subseteq \alpha\text{Int}(f^i(\mu))$. Since $f$ is one-one, $\alpha\text{Int}(f^i(\mu)) = \alpha\text{Int}(\mu)$. This shows that, $f^i(s\text{Int}(f(\mu))) \subseteq \alpha\text{Int}(\mu)$. Further since $f$ is onto we have, $s\text{Int}(f(\mu)) = f^i(s\text{Int}(f(\mu))) \subseteq f(\alpha\text{Int}(\mu))$. Thus $s\text{Int}(f(\lambda)) \subseteq f(\alpha\text{Int}(\mu))$.

Conversely let $\lambda$ be a g-fuzzy semi-open set in $Y$. Then $s\text{Int}(f(\lambda)) = \lambda$. Now $f^i(\lambda)$ is a g-fuzzy set in $X$, from hypothesis, $f(\alpha\text{Int}(f^i(\lambda))) \subseteq s\text{Int}(f(f^i(\lambda)))$. Since $f$ is onto, $s\text{Int}(f(f^i(\lambda))) = \alpha\text{Int}(\lambda)$. Therefore $f(\alpha\text{Int}(f^i(\lambda))) \subseteq \lambda$. Further since $f$ is one-one, $\alpha\text{Int}(f^i(\lambda)) = f^i(f(\alpha\text{Int}(f^i(\lambda))) \subseteq f^i(\lambda)$. As $\alpha\text{Int}(f^i(\lambda)) \subseteq f^i(\lambda)$, we deduce that $f^i(\lambda) = \alpha\text{Int}(f^i(\lambda))$. Thus $f^i(\lambda)$ is a fuzzy $\alpha$-open set in $X$. Hence $f : X \to Y$ is g-fuzzy strongly $\alpha$-continuous map.

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